Module 5
Circular Functions and Trigonometry

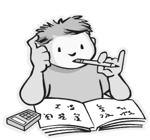
What this module is about

This module is about trigonometric equations and proving fundamental identities. The lessons in this module were presented in a very simple way so it will be easy for you to understand solve problems without difficulty. Your knowledge in previous lessons would be of help in the process

What you are expected to learn

This module is designed for you to:

1. state the fundamental identities
2. prove trigonometric identities
3. state and illustrate the sum and cosine formulas of cosine and sine
4. determine the sine and cosine of an angle using the sum and difference formulas.
5. solve simple trigonometric equations



How much do you know

A. Answer the following:

1. Which of the following does not equal to 1 for all A in each domain?

   a. $\sin^2 A + \cos^2 A$ b. $\sec^2 A - \cos^{2 A}$

   c. $\sin A \sec A$ d. $\tan A \cot A$

2. Simplify $\cos^2 A \sec A \csc A$

3. If sin $\propto$ = $\frac{12}{13}$ and cos β = $\frac{4}{5}$, where $\propto$ and β are both in the first quadrant, find the values of cos ($\propto$ + β).

4. Sec A is equal to

a. cos A b. sin A c. $\frac{1}{\cos A}$ d. $\frac{1}{\sin A}$.

5. Express $\frac{1-\csc B}{\cot B}$ in terms of cos B and Sin A.

a. cos B – sinB b. $\frac{1-\sin B}{\cos B}$ c. sin B – cos B d. $\frac{\sin B-1}{\cos B}$

6. Simplify $\frac{\cos\phi}{\sin\phi\cot\phi}$.

a. 1 b. tanϕ c. –csc ϕ d. -1

7. Multiply and simplify (1 – $\cos^2$ t) (1 + $\tan^2$ t).

8. Express tan B (sin B + cot B + cos B) in terms of sec B.

9. Compute sin $\frac{5\pi}{12}$ from the function of $\frac{\pi}{4}$ and $\frac{\pi}{6}$.

10. Solve the equation cos A – 2sin A cos A = 0.

What you will do

Lesson 1

Fundamental Trigonometric Identities

To be able to simplify trigonometric expressions and solve trigonometric equations, you must be able to know the fundamental trigonometric identities.

The Eight Fundamental Identities:

A. Reciprocal Relations

1. $\sec\theta = \dfrac{1}{\cos\theta}$

2. $\csc\theta = \dfrac{1}{\sin\theta}$

3. $\cot\theta = \dfrac{1}{\tan\theta}$

B. Quotient Relations

4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

5. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

C. Pythagorean Relations

6. $\cos^2\theta + \sin^2\theta = 1$

7. $1 + \tan^2\theta = \sec^2\theta$

8. $\cot^2\theta + 1 = \csc^2\theta$

With the aid of this identities, you may now simplify trigonometric expressions.

Examples:

Perform the indicated operation.

a. (1 – sin x) (1 + sin x)

$= 1 - \sin^2 x$ — Product of sum & difference of 2 terms

$= \cos^2 x$ — Since, $\cos^2 \theta + \sin^2 \theta = 1$, then $1 - \sin^2 x = \cos^2 x$

b. (sec A – 1) (sec A + 1)

$= \sec^2 A - 1$
$= \tan^2 A$ — Pythagorean Relation no. 2

c. $\tan \theta (\cot \theta + \tan \theta)$

$= \tan \theta \cot \theta + \tan^2 \theta$
$= 1 + \tan$
$= \sec^2 \theta$

$\tan \theta \cot \theta = 1$, because

$\tan \theta = \frac{\sin\theta}{\cos\theta}$ $\quad$ $\cot \theta = \frac{\sin\theta}{\cos\theta}$

$\tan \theta \cot \theta = (\frac{\sin\theta}{\cos\theta})(\frac{\sin\theta}{\cos\theta}) = 1$

d. cos x (sec x - cos x)

$= \cos x \sec x - \cos^2 x$
$= 1 - \cos^2 x$
$= \sin^2 x$

$\cos x \sec x = 1$, because $\sec x = \frac{1}{\cos x}$

therefore, $\cos x \sec x = \cos x (\frac{1}{\cos x}) = 1$

e. cos B + $\frac{\sin^2 B}{\cos B}$

= $\frac{\cos^2 B + \sin^2 B}{\cos B}$ cos B, Least common denominator

= $\frac{1}{\cos B}$ Identity C. 6

= sec B Identity A. 1

Simplify the following expressions to a single function.

a. $\cos^2 A \tan^2 A = \sin^2 A$

$\cos^2 A \left(\frac{\sin^2 A}{\cos^2 A} \right) = \sin^2 A$

b. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

= $\sin^2 x + \cancel{2\sin x \cos x} + \cos^2 x + \sin^2 x - \cancel{2 \sin x \cos x} + \cos^2 x$

= $\sin^2 x + \cos^2 x + \sin^2 x + \cos^2 x$

= 2

since, $\cos^2 \theta + \sin^2 \theta = 1$

c. cot B sec B sin B = 1

since, $\cot \theta = \frac{\cos\theta}{\sin\theta}$ and $\sec \theta = \frac{1}{\cos\theta}$

then, $\left(\frac{\cancel{\cos\theta}}{\cancel{\sin\theta}}\right)\left(\frac{1}{\cancel{\cos\theta}}\right)\cancel{\sin}\,\theta = 1$

d. $\csc A - \csc A \cos^2 A = \sin A$

$= \csc A (1 - \cos^2 A)$ Factor csc A

$= \csc A (\sin^2 A)$ Identity C. 6

$= \frac{1}{\sin\theta} (\sin^2 A)$ Identity A. 2

$= \sin A$ Cancellation

e. $\cos^3 B + \cos B \sin^2 B$

$= \cos B (\cos^2 B + \sin^2 B)$ Factor cos B

$= \cos B (1)$ Identity C. 6

$= \cos B$

You are now ready to prove identities. In this lesson, you will prove that one side of the equation is equal to the other side. You can work on either of the two sides to verify the expressions are equal or you can work on both equations to arrive at an equal statement.

Suggested Steps in Proving Identities

1. Start with the more complicated side and transform it into the simpler side.
2. Try algebraic operations such as multiplying, factorings, splitting single fractions and so on.
3. If other steps fail, express each function in terms of sine and cosine functions and then perform appropriate algebraic operations.
4. At each step, keep the other side of the identity in mind. This often reveals what one should do in order to get there.

Examples:

a. Prove: cos B + tan B sin B = sec B

Solution:

Generally, we start with the more complicated side and transform it into the other side using fundamental identities, algebra or other establish identities.

$\cos B + \dfrac{\sin B}{\cos B}$ (sin B) substituting $\dfrac{\sin B}{\cos B}$ to tan B

$\dfrac{\cos\ B + \sin^2 B}{\cos B}$ addition of fractions

$\dfrac{1}{\cos B}$ = sec B identity A. 1

b. Prove that sec A - tan A sin A = cos A

Solution:

sec A - tan A sin A = cos A
sec A - tan A sin A =

$\dfrac{1}{\cos A} - \dfrac{\sin A}{\cos A}$ sin A = Substitute $\dfrac{\sin A}{\cos A}$ to tan A

$\dfrac{1}{\cos A} - \dfrac{\sin^2 A}{\cos A}$ = Subtraction of fraction

$\dfrac{1-\sin^2 A}{\cos A}$ = Identity C. 6

$\dfrac{\cos^2 A}{\cos A}$ = Division of fraction

cos A = cos A

c Prove: $\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = 2\sec^2\theta$

Solution:

In this problem, let us concentrate on the left side because the left side looks more complicated than the right side. By adding the two fractions, we get:

$$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{1+\sin\theta+1-\sin\theta}{(1-\sin\varpi)(1+\sin\theta)}$$

$$= \frac{2}{1-\sin^2\theta}$$

$$= \frac{2}{\cos^2\theta}$$

$$2\sec^2\theta = 2\sec^2\theta$$

d. Prove: $\sec^4 A - \sec^2 A = \tan^2 A + \tan^4 A$

Solution:

$$\sec^4 A - \sec^2 A = \sec^2 A(\sec^2 A - 1)$$

$$= \sec^2 A \tan^2 A$$

$$= 1 + \tan^2 A\ (\tan^2 A)$$

$$= \tan^2 A + \tan^4 A$$

e. Prove: $\sin\theta + \cos\theta + \frac{\sin\theta}{\cot\theta} = \sec\theta + \csc\theta - \frac{\cos\theta}{\tan\theta}$

For this example, we will work on both sides of the equation.

Let us first work on the left-hand side of the equation:

$$\sin\theta + \cos\theta + \frac{\sin\theta}{\cot\theta} = \sin\theta + \cos\theta + \frac{\sin\theta}{\frac{\cos\theta}{\sin\theta}}$$

$$= \sin\theta + \cos\theta + \frac{\sin^2\theta}{\cos\theta}$$

$$= \sin\theta + \cos\theta + \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \sin\theta + \cos\theta + \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$

$$= \sin\theta + \cos\theta + \sec\theta - \cos\theta$$

$$= \sin\theta + \sec\theta$$

Now, work on the right-hand side of the eqution:

$$\sec\theta + \csc\theta - \frac{\cos\theta}{\tan\theta} = \sec\theta + \csc\theta - \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \sec\theta + \csc\theta - \frac{\cos^2\theta}{\sin\theta}$$

$$= \sec\theta + \csc\theta - \frac{1-\sin^2\theta}{\sin\theta}$$

$$= \sec\theta + \csc\theta - \frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}$$

$$= \sec\theta + \cancel{\csc}\,\theta - \cancel{\csc}\,\theta - \sin\theta$$

$$= \sec\theta - \sin\theta$$

Therefore:

$$\sec\theta - \sin\theta = \sec\theta - \sin\theta$$

f. Prove the identity

$$\frac{1+\cos B}{1-\cos B} + \frac{1+\sin B}{1-\sin B} = \frac{2\cos B-\csc B}{\cot B-\cos B-\csc B+1}$$

Begin working with the left side,

$$\frac{1+\cos B}{1-\cos B} + \frac{1+\sin B}{1-\sin B} = \frac{(1+\cos B)(1-\sin B)+(1+\sin B)(1-\cos B)}{(1-\cos B)(1-\sin B)}$$

$$= \frac{(1+\cos B)-\sin B-\cos B\sin B+1+\sin B-\cos B-\sin B\cos B}{(1-\cos B)(1-\sin B)}$$

$$= \frac{2-2coB\sin B}{1-\cos B-\sin B+\cos B\sin B}$$

Now work on the right side of the equation.

$$\frac{2\cos B-\csc B}{\cot B-\cos B-\csc B+1} = \frac{2\cos B-\dfrac{2}{\sin B}}{\dfrac{\cos B}{\sin B}-\cos B-\dfrac{1}{\sin B}+1}$$

$$= \frac{\dfrac{2\cos B\sin B - 2}{\sin B}}{\dfrac{\cos B - \cos B\sin B - 1 + \sin B}{\sin B}}$$

$$= \frac{2-2\cos B\sin B}{\cos B-\cos B\sin B-1=\sin B}$$

$$= \frac{2-2\cos B\sin B}{1-\cos B-\sin B+\cos B\sin B}$$

Try this out

A. Perform the indicated operation and simplify.

1. $(\sin\theta + \cos\theta)^2$

2. $(\cot B + \csc B)(\cot B - \csc B)$

3. $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta}$

4. $\tan\theta - \dfrac{\sec^2\theta}{\tan\theta}$

B. Factor each expression and simplify

1. $\tan^2 A - \tan^2 A \sin^2 A$

2. $\sec^2 B \tan^2 B + \sec^2 B$

3. $\tan^4 A + 2\tan^2 A + 1$

C. Prove the following identities.

1. $\sin x \cos x \cot x = \cos^2 x$

2. $1 - \sin t \cos t \cot t = \sin^2 t$

3. $\sin A \cos A \tan A + \sin A \cos A \cot A = 1$

4. $\sec B \cot B = \csc B$

5. $\dfrac{1+\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{\cos\theta + 1}{\sin\theta\cos\theta}$

6. $\dfrac{1+\tan\theta}{1+\cot\theta} = \dfrac{\sec\theta}{\csc\theta}$

7. $\dfrac{\sin\theta+\cos\theta}{\sec\theta+\csc\theta} = \dfrac{\sin\theta}{\sec\theta}$

Lesson 2

The Sum and Difference of Sine and Cosine

The sum and difference identities for sine and cosine can be used to find the exact values of the sine and cosine of angles which is not exact.

The Sum and Difference Identities:

cos (A - B) = cos A cos B + sin A cos A

cos (A + B) = cos A Cos B - sin A sin B

sin (A + B) = sin A cos B + sin B cos A

sin (A – B) = sin A cos B – sin B cos A

The sum and difference identities for sine and cosine can be used to find the exact values of the sine and cosine of angles that are not special angles.

Example 1:

a. Find the exact value of sin 15°

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

b. find the exact value of cos $\frac{5\pi}{12}$.

$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 2:

Consider sin A = $\frac{12}{13}$ with P(A) in Q1 and cos B = $\frac{3}{5}$ with P(B)in Q1.

Find: a. sin (A + B) b. cos (A + B)

Solution:

Since P(A) and P(B) are both in Q1, P(A + B) is either in the first or second quadrant. While, cosine is positive in the first quadrant and negative in the second, it will suffice to find sin (A + B) and cos (A + B).

We get the value of cos A and sin B by using the Pythagorean relation,

$$\cos^2 A + \sin^2 B = 1$$

Substitute $\frac{12}{13}$ for sin A and $\frac{3}{5}$ for cos B.

$$\cos^2 A + \sin^2 A = 1$$

$$\cos^2 A + \left(\frac{12}{13}\right)^2 = 1$$

$$\cos^2 A + \frac{144}{169} = 1$$

$$\cos^2 A = 1 - \frac{144}{169}$$

$$\cos^2 A = \frac{25}{169}$$

$$\sqrt{\cos^2 A} = \sqrt{\frac{25}{169}}$$

$$\cos A = \pm\frac{5}{13}$$

Take $\cos A = \frac{5}{13}$ since P(A) is in Q1.

Substitute $\frac{3}{5}$ for cos B.

$$\cos^2 B + \sin^2 B = 1$$

$$\left(\frac{3}{5}\right)^2 + \sin^2 B = 1$$

$$\frac{9}{25} + \sin^2 B = 1$$

$$\sin^2 B = 1 - \frac{9}{25}$$

$$\sin^2 B = \frac{16}{25}$$

$$\sqrt{\sin^2 B} = \sqrt{\frac{16}{25}}$$

$$\sin B = \pm\frac{4}{5}$$

Take $\sin B = \frac{4}{5}$

a. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{12}{13}\right)\left(\frac{3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$\sin(A + B) = \frac{56}{65}$$

b. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$

$$= \frac{15}{65} - \frac{48}{65}$$

$$\cos(A + B) = \frac{-33}{65}$$

Example 3:

If $\sin A = \frac{-1}{3}$, $180° < \angle A < 270°$ and $\cos B = \frac{1}{5}$, $270° < \angle B < 360°$.

Evaluate:

a. $\cos(A - B)$ b. $\sin(A – B)$

Solution:

First, you have to get the values of cos A and sin B. We can get these values by using the Pythagorean relation.

Since, $180° < \angle A < 270°$, $\angle A$ is in quadrant III and $\angle B$ is in quadrant IV.

$$\cos^2 A + \sin^2 A = 1$$

$$\cos^2 A + \left(\frac{-1}{3}\right)^2 = 1$$

$$\cos^2 A + \frac{1}{9} = 1$$

$$\cos^2 A = \frac{8}{9}$$

$$\cos A = \pm\frac{2\sqrt{2}}{3}$$

$$\cos A = -\frac{2\sqrt{2}}{3}, \text{ since } \angle A \text{ is in Q III.}$$

Now, find the value of cos B:

$$\cos B = \frac{1}{5}, \quad 270° < \angle B < 360°.$$

$$\cos^2 B + \sin^2 B = 1$$

$$\left(\frac{1}{5}\right)^2 + \sin^2 B = 1$$

$$\sin^2 B = \frac{24}{25}$$

$$\sin B = \pm\frac{2\sqrt{6}}{5}$$

$$\sin B = -\frac{2\sqrt{6}}{5}, \text{ since } \angle B \text{ is Q IV.}$$

We are now ready to evaluate: a. cos(A - B)

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{5}\right) + \left(\frac{-1}{3}\right)\left(-\frac{2\sqrt{6}}{5}\right)$$

$$= \frac{-2\sqrt{2}}{15} + \frac{2\sqrt{6}}{15}$$

$$\cos(A - B) = \frac{-2\sqrt{2} + 2\sqrt{6}}{15}$$

b. sin(A – B) = sin A cos B - cos A sin B

$$= \left(\frac{-1}{3}\right)\left(\frac{1}{5}\right) - \left(-\frac{2\sqrt{2}}{3}\right)\left(-\frac{2\sqrt{6}}{5}\right)$$

$$= \frac{-1}{15} - \frac{4\sqrt{12}}{15}$$

$$= \frac{-1}{15} - \frac{8\sqrt{3}}{15}$$

$$\sin(A - B) = \frac{-1 - 8\sqrt{3}}{15}$$

Try this out

A. Express each measure as the sum or difference of two special angle measures.

1. 105°
2. 135°
3. 225°
4. 315°
5. $\frac{\pi}{6}$
6. $\frac{7\pi}{12}$

B. Evaluate by using special angle measures and a sum or difference identity.

1. cos 105°
2. sin 135°
3. sin 225°
4. cos 315°
5. cos $\frac{\pi}{6}$
6. sin $\frac{7\pi}{12}$
7. sin $\frac{8\pi}{3}$
8. sin $\frac{4\pi}{3}$
9. sin 300°
10. cos 240°

C. Evaluate sin(A + B) and sin(A - B) given the following:

1. $\sin A = \frac{4}{5}, 90^\circ \langle \angle A \langle 180^\circ$ and $\cos B = \frac{12}{13}, 180^\circ \langle \angle B \langle 270^\circ$

2. $\sin A = \frac{3}{5}, 90^\circ \langle \angle A \langle 180^\circ$ and $\cos B = -\frac{12}{13}, 180^\circ \langle \angle B \langle 270^\circ$

Lesson 3

Trigonometric Equations

A trigonometric equation is an equation which involves some trigonometric functions of the variable. The rules in solving algebraic equations also applies to the solution of trigonometric equations.

Here, are some pointers for you to remember:

1. Adding the same expression to both members of an equation produces an equivalent equation.

2. Multiplying each member of an equation by the same expression produces an equivalent equation.

3. Replacing any expression is an equation by another expression representing the same real number produces an equivalent equation.

Let us review your knowledge on solving algebraic equations:

Solve for the value of x in the following equations.

1. $x^2 - 7x + 12 = 0$

$(x - 4)(x - 3) = 0$

$x - 4 = 0$ or $x - 3 = 0$

$x = 4$ or $x = 3$

2. $6x^2 - 5x = -1$

$6x^2 - 5x + 1 = 0$

$(3x - 1)(2x - 1) = 0$

$x = \frac{1}{3}$ or $x = \frac{1}{2}$

You can now apply these procedure to solve trigonometric equations.

Examples:

a. Solve: $\sin^2 x + 3 \sin x + 2 = 0, \ 0 \le x \langle \ 2\pi$

$$\sin^2 x + 3 \sin x + 2 = 0$$

$$(\sin x + 2)(\sin x + 1) = 0$$

$$\sin x + 2 = 0 \quad \text{or} \quad \sin x + 1 = 0$$

$$\sin x = -2 \quad \text{or} \quad \sin x = -1$$

You have to reject -2 since, $-1 \le x \le 1$

$\sin x = -1$ and $x = \frac{3\pi}{2}$ since $\sin \frac{3\pi}{2} = -1$.

Therefore, the solution within the specified interval is $\frac{3\pi}{2}$.

$\frac{3\pi}{2}$ is called a **primary solution** because it is a solution within the given interval. All angles that are coterminal with the angle that is a primary solution would also be a solution. These solutions differ from the primary solution by integral multiples of the period of the function and are called **general solutions** of the equation. The general solution of the equation is

$$x = \frac{3\pi}{2} + \pi\text{, where k is an integer.}$$

Forming the General Solution

For equations of the form

$\sin x = k \qquad \cos x = k \qquad \csc x = k \qquad \sec x = k$

find all solutions in the interval $0 \le x \le k\pi$ and add $2k\pi$ to each to form the general solution..

For equations of the form

$\tan x = k \qquad \cot x = k$

find all solutions in the interval $0 \le x \le k\pi$ and add $k\pi$ to each to form the general solution.

When a trigonometric equation contains more than one function, transform it into an equation containing only one trigonometric function. Use the identities and substitute.

b. Solve: $5 \sec^2 x + 2 \tan x = 8, \; 0° \le x \le 360°$

$$5 \sec^2 x + 2 \tan x = 8$$

$5(\tan^2 x + 1) + 2 \tan x - 8 = 0$ substitute $\tan^2 x + 1$ for $\sec^2 x$

$$5 \tan^2 x + 5 + 2 \tan x - 8 = 0$$

$$5 \tan^2 x + 2 \tan x - 3 = 0$$

$$(5 \tan x - 3)(\tan x + 1) = 0$$

$$\tan x = \frac{3}{5} \quad \text{or} \quad \tan x = -1$$

$$\tan x = 0.6 \quad \text{or} \quad \tan x = -1$$

$$x = 31° \quad \text{or} \quad x = 135°$$

$$x = 211° \quad \text{or} \quad x = 315°$$

Since the tangent functions has a period of 180°, the solutions within the specified interval are 31°, 135 °, 211° and 315 °.

c. Solve $\cot x \cos^2 x = 2 \cot x$

$$\cot x \cos^2 x = 2 \cot x$$

$$\cot x \cos^2 x - 2 \cot x = 0$$

$$\cot x (\cos^2 x - 2) = 0$$

$$\cot x = 0 \quad \text{or} \quad \cos^2 x - 2 = 0$$

$$x = \frac{\pi}{2} \qquad \cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}$$

No solution is obtained from cos x $= \pm\sqrt{2}$, since $\pm\sqrt{2}$ is outside the cosine function. The complete solution is

$$x = \frac{\pi}{2} + k\pi, \text{ where k is an integer.}$$

d. Solve cos A + sin A = 1

Solve for cos A. Then square both sides.

$$\cos A + \sin A = 1$$

$$\cos A = 1 - \sin A$$

$$\cos^2 A = (1 - \sin A)^2$$

$$\cos^2 A = 1 - 2\sin A + \sin^2 A$$

$$1 - \sin^2 A = 1 - 2\sin A + \sin^2 A$$

$$2\sin A - 2\sin^2 A = 0$$

$$2\sin A\ (1 - \sin A) = 0$$

$$2\sin A = 0 \quad \text{or} \quad 1 - \sin A = 0$$

2 sin A = 0 implies that sin A = 1, and A = 2π.

1 - sin A = 0 implies that sin A = 1, A = $\frac{\pi}{2}$

Try this out

A. Find all values of θ between 0 and 2π that satisfy each of the following equations.

1. $\sin\theta = \frac{1}{2}$

2. $\cos\theta = \frac{-\sqrt{3}}{2}$

3. $\tan\theta = -1$

4. $\cot\theta = -\sqrt{3}$

5. tan θ = $\frac{1}{-\sqrt{3}}$

6. cos θ = $\frac{-1}{2}$

7. sin θ = $\frac{\sqrt{2}}{2}$

8. sin θ = $\frac{-\sqrt{3}}{2}$

9. cos θ = $-\frac{\sqrt{2}}{2}$

10. sec θ = 2

B. Find the solutions in each of the following in the interval 0, 2π

1. $2 \sin^2 x = 1$
2. $\tan x (\tan x - 1) = 0$
3. $3 \sec^2 x - 4 = 0$
4. $2 \cos^2 x \tan x - \tan x = 0$
5. $2 \cot^2 x + \csc^2 x = 2$
6. $1 + \cos A = 3 \cos A$
7. $2 \tan^2 B + \sec^2 B = 2$
8. $\cos A = \sin A$
9. $\sin^2 A + 2\sin A + 1 = 0$
10. $\cos^2 x = \cos x$

Let's summarize

The Eight Fundamental Identities

A. Reciprocal Relations

1. $\sec\theta = \dfrac{1}{\cos\theta}$

2. $\csc\theta = \dfrac{1}{\sin\theta}$

3. $\cot\theta = \dfrac{1}{\tan\theta}$

B. Quotient Relations

4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

5. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

C. Pythagorean Relations

6. $\cos^2\theta + \sin^2\theta = 1$

7. $1 + \tan^2\theta = \sec^2\theta$

8. $\cot^2\theta + 1 = \csc^2\theta$

The Sum and Difference Identities

1. cos (A - B) = cos A cos B + sin A cos A

2. cos (A + B) = cos A Cos B - sin A sin B

3. sin (A + B) = sin A cos B + sin B cos A

4. sin (A – B) = sin A cos B – sin B cos A

The sum and difference identities for sine and cosine can be used to find the exact values of the sine and cosine of angles that are not special angles.

The General Solution:

For equations of the form,

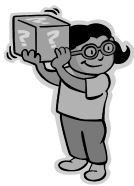
$\sin x = k$ $\qquad \cos x = k$ $\qquad \csc x = k$ $\qquad \sec x = k$

find all solutions in the interval $0 \le x \le k\pi$ and add $2k\pi$ to each to form the general solution.

For equations of the form,

$\tan x = k \qquad \cot x = k$

find all solutions in the interval $0 \le x \le k\pi$ and add $k\pi$ to each to form the general solution.



What have you learned

Answer the following:

1. What is cos x if $\sin x = \frac{3}{5}$ and the terminal side of $\angle X$ is in Q1.
2. Express $\cos B + \frac{\sin^2 B}{\cos B}$ in simplest form.
3. Factor and simplify. $\cos^3 A - \cos A$
4. Prove: $\frac{\cos A \csc A}{\cot A} = 1$
5. Simplify cot B sec B sin B to a single function.
6. Rationalize the denominator and simplify. $\frac{\sin^2 \theta}{1 - \cos\theta}$.
7. Write cos 330° as sum and difference of 2 special angles.
8. If $\angle A$ is a fourth-quadrant angle, $\sin A = \frac{-8}{17}$, $\angle B$ is a 3rd-quadrant angle, and $\cos B = \frac{-5}{13}$. Find the exact value of cos(A + B) and cos (A – B).
9. Solve for x, $0 \le x \le \frac{\pi}{2}$. $1 - 2\sin A = 0$
10. Prove: $\tan\theta(\sin\theta + \cos\theta)^2 + (1 - \sec^2\theta)\cot\theta = 2\sin^2\theta$

Answer Key

How much do you know

1. c 2. cot A 3. $\frac{228}{4225}$ 4. c 5. d 6. a 7. $\tan^2\theta$

8. $\frac{\sin B}{\cos B}$ (sin B) + $(\frac{\cos B}{\sin B})(\frac{\sin B}{\cos B})$ cos A

9. $\frac{\sqrt{6} - \sqrt{2}}{4}$

$\frac{\sin^2 B}{\cos B}$ + cos B = $\frac{\sin^2 B + \cos^2 B}{\cos B}$ = $\frac{1}{\cos B}$ = sec B

10. $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Try this out

Lesson 1

A 1. $1 + 2\sin\theta\cos\theta$ 2. $\frac{\cos^2\theta - 1}{\sin^2\theta}$ 3. $\frac{1+2\sin\theta}{\cos\theta(1+\sin\theta)}$ 4. $\frac{-1}{\tan\theta}$ or $\frac{-\cos\theta}{\sin\theta}$

B 1. $\tan^2 A\,(1 - \sin^2 A)$
$= \tan^2 A\,(\cos^2 A)$
$= \frac{\sin^2 A}{\cos^2 A}(\cos A)$
$= \sin^2 A$

2. $\sec^2 B \tan^2 B + \sec^2 B$
$= \sec^2 B\,(\tan^2 B + 1)$
$= \sec^2 B\,(\sec^2 B)$
$= \sec^4 B$

3. $(\tan^2 B + 1)^2$
$= \sec^4 A$

C1. sin x cos x cot x = cos x
sin x cos x $\frac{\cos x}{\sin x}$ = $\cos^2 x$

2. 1 - sin t cos t cot t = $\sin^2 t$
= 1 - sin t cos t $(\frac{\cos t}{\sin t})$ = $1 - \cos^2 t = \sin^2 t$

3. sin A cos A tan A + sin A cos A cot A = 1

= sin A cos A $(\frac{\sin A}{\cos A})$ + sin A cos A $(\frac{\cos A}{\sin A})$ = $\sin^2 A + \cos^2 A = 1$

4. sec B cot B = csc B $(\frac{1}{\cos B})(\frac{\cos B}{\sin B})$ = $\frac{1}{\sin B}$ = csc B

5. $\frac{1+\cos\theta}{\sin\theta}$ + $\frac{\sin\theta}{\cos\theta}$ = $\frac{\cos\theta(1+\cos\theta)+\sin^2\theta}{\sin\theta\cos\theta}$ = $\frac{\cos\theta+\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}$

= $\frac{\cos\theta+1}{\sin\theta\cos\theta}$

6. $\frac{1+\tan\theta}{1+\cot\theta}$ = $\frac{\cos\theta+\sin\theta}{\cos\theta}$. $\frac{\sin\theta}{\sin\theta+\cos\theta}$

= $\frac{\sin\theta}{\cos\theta}$ = $\frac{\sec\theta}{\csc\theta}$

7. $\frac{\sin x+\cos x}{\sec x+\csc x}$ = $\frac{\sin x+\cos x}{\frac{\sin x+\cos x}{cox\sin x}}$ = sin x + cos x ($\frac{\cos x\sin x}{\sin x+\cos x}$) = cos x sin x = $\frac{\sin\theta}{\sec\theta}$

Lesson 2

A. 1. (60° + 45°) 2. (90° + 45°) 3. (180° + 45°) 4. (270° + 45°)

5. $\frac{\pi}{3}$ - $\frac{\pi}{6}$ 6. $\frac{\pi}{4}$ + $\frac{\pi}{3}$

B. 1. $\frac{\sqrt{2} - \sqrt{6}}{4}$ 2. $\frac{\sqrt{2}}{2}$ 3. $\frac{\sqrt{2}}{2}$ 4. -$\frac{\sqrt{2}}{2}$ 5. $\frac{\sqrt{3} - 1}{4}$

6. $\frac{\sqrt{2} + \sqrt{6}}{4}$ 7. $\frac{\sqrt{3}}{2}$ 8. - $\frac{\sqrt{3}}{2}$ 9. $\frac{-1}{2}$ 10. - $\frac{\sqrt{3}}{2}$

C. 1. cos A = $\frac{-3}{5}$ sin B = $\frac{-5}{13}$ sin (A + B) = $\frac{63}{65}$ sin (A - B) = $\frac{-7}{13}$

2. cos A = $\frac{-4}{5}$ sin B = $\frac{-5}{13}$ sin (A + B) = $\frac{1}{13}$ sin (A - B) = $\frac{-44}{65}$

Lesson 3

A. 1. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ 2. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ 3. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ 4. $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$

5. $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ 6. $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ 7. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ 8. $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$

9. $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ 10. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

B. 1. $x = \frac{\pi}{4}, \frac{7\pi}{4}$ 2. $x = \frac{\pi}{4}, \frac{5\pi}{4}$ 3. $x = \frac{\pi}{6}, \frac{11\pi}{6}$

4. $x = 2\pi$ 5. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ 6. $\frac{\pi}{3}, \frac{5\pi}{3}$

7. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ 8. $x = \frac{\pi}{4}, \frac{5\pi}{4}$

9. $x = \frac{3\pi}{2}$ 10. $x = 0, 2\pi$

What have you learned

1. $\cos x = \frac{4}{25}$

2. $\frac{\cos^2 B + \sin^2 B}{\cos B} = \frac{1}{\cos B} = \sec B$

3. cos A (cos A + 1)(cos A – 1)

4. $\dfrac{\cos A \dfrac{1}{\sin A}}{\dfrac{\cos A}{\sin A}}$ = $\dfrac{\cos A}{\sin A}$ x $\dfrac{\sin A}{\cos A}$ = 1

5. $\dfrac{\sin\theta}{\cos\theta}$ x cos θ - $\dfrac{\cos\theta}{\sin\theta}$ x sin θ = sin θ - cos θ

6. $\dfrac{\sin^2\theta + \cos\theta}{\sin^2\theta}$

7. cos (270° + 60°) = $\dfrac{\sqrt{3}}{2}$

8. cos A = $\dfrac{15}{17}$ sin A = $\dfrac{-12}{13}$ cos (A + B) = $\dfrac{-171}{221}$

9. X = $\dfrac{\pi}{6}$, $\dfrac{\pi}{3}$

10. sin θ + 2sin^2 θ + $\dfrac{1-\sin^2\theta - 1}{\sin\theta}$ = $\dfrac{\sin\theta + 2\sin^3\theta - \sin^2\theta}{\sin\theta}$ = 2 sin^2 θ