Module 4
Circular Functions and Trigonometry

What this module is about

This module is about the properties of the graphs of a circular functions. You will learn how the graphs of circular function look like and how they behave in the coordinate plane.

What you are expected to learn

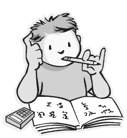
This module is designed for you to:

1. describe the properties of the graphs of the functions:

    - sine
    - cosine
    - tangent

2. graph the sine, cosine and tangent functions.

3. solve trigonometric equations.



How much do you know

1. What is The period of the sine function y = sin x?

    a. 2π b. $\frac{\pi}{2}$ c. π d. $\frac{3\pi}{2}$

2. What is the amplitude of a cosine function y = cos x?

    a. -2 b. -1 c. 2 d. 1

3. What is the value of $y = 4 \sin \frac{1}{2} x$, if $x = \frac{5\pi}{3}$?

4. What is the value of y = 2 sin x, if x = $\frac{\pi}{4}$?

Given the following functions, identify the amplitude of :

5. y = 2 cos x

6. y = $\frac{3}{4}$ sin x

7. y = -2 sin $\frac{3}{2}$ x

Given the following functions, determine the period of:

8. y = 2 cos $\frac{1}{2}$ x

9. y = 4 sin $\frac{2}{3}$ x

10. y = sin $\frac{1}{5}$x

11. Which of the following are zeros of y = tan Θ for the interval 0 ≤ Θ ≤ 2π?

a. 0, π and 2π

b. $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

c. $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

d. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

12. In which of the following intervals is the cosine function decreasing over the interval [0, 2π]?

a. [0, π]

b. $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$

c. [π, 2π]

d. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

13. solve for the solution set of sin x – 1 = 0 in the interval $0 \leq \theta \leq 2\pi$.

What you will do

Lesson 1

Graphs of Sine, Cosine and Tangent

Circular functions can also be graphed just like the other functions you have learned before. The difference is that the graphs of circular functions are *periodic*. A function is said to be periodic if the dependent variable y takes on the same values repeatedly as the independent variable x changes.

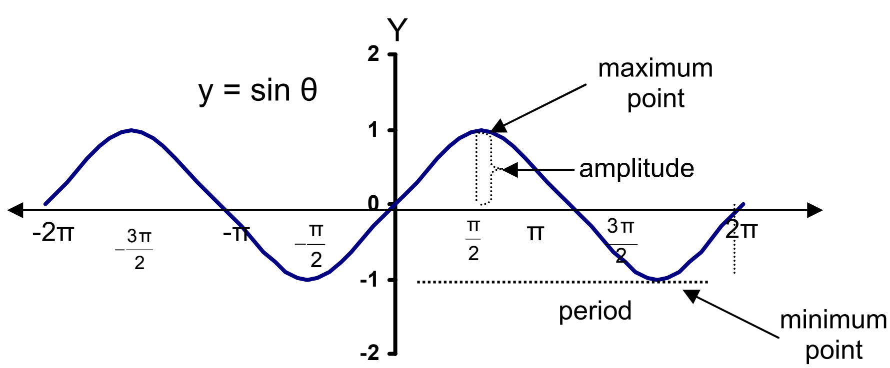
Observe the changes in the values of y = sin θ and y = cos θ for arc lengths from -2π to 2π.

θ	$-\frac{3\pi}{2}$ to -2π	$-\pi$ to $-\frac{3\pi}{2}$	$-\frac{\pi}{2}$ to $-\pi$	0 to $-\frac{\pi}{2}$	0 to $\frac{\pi}{2}$	$\frac{\pi}{2}$ to π	π to $\frac{3\pi}{2}$	$\frac{3\pi}{2}$ to 2π
sin θ	1 to 0	0 to 1	-1 to 0	0 to -1	0 to 1	1 to 0	0 to -1	-1 to 0
cos θ	0 to 1	-1 to 0	0 to -1	1 to 0	1 to 0	0 to -1	-1 to 0	0 to 1

Using the arc length, θ, as the independent variable and y = sin θ and y = cos θ as the dependent variables, the graphs of the sine and cosine functions can be drawn.

Below is the graph of y = sin θ for -2π ≤ θ ≤ 2π. This was done by plotting the ordinates on the y-axis and the arc lengths on the x-axis.

Observe the properties of this graph.

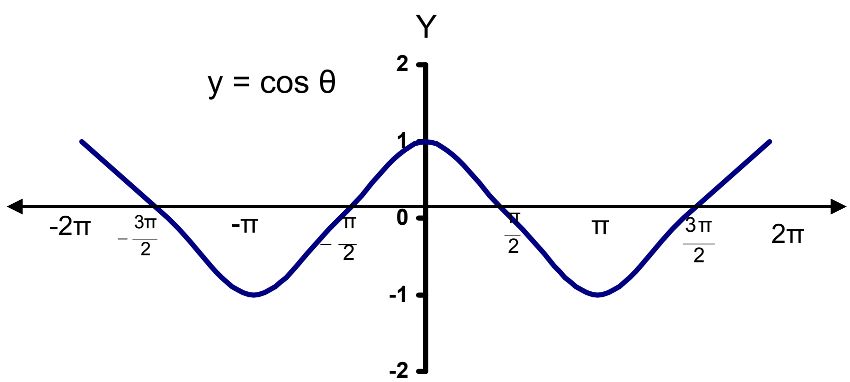


You can see that the graph is a curve. Call this the sine curve. Observe that the graph contains a cycle. One complete cycle is the interval from -2π to 0 and another cycle is the interval from 0 to 2π. This is called the period of the curve. Hence, the period of y = sin θ is 2π.

The amplitude of the graph of y = sin θ is 1. The amplitude is obtained by getting the average of the maximum value and the minimum value of the function. The maximum point is $\left(\frac{\pi}{2}, 1\right)$ and the minimum point is $\left(\frac{3\pi}{2}, -1\right)$ for the interval $[0,\ 2\pi]$. The graph crosses the x-axis at $(0,0)$, $(\pi,0)$, and $(2\pi,0)$ for the interval $[0,\ 2\pi]$. Observe also that the sine graph is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π, and decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ for the interval [0, 2π].

The Graph of Cosine Function

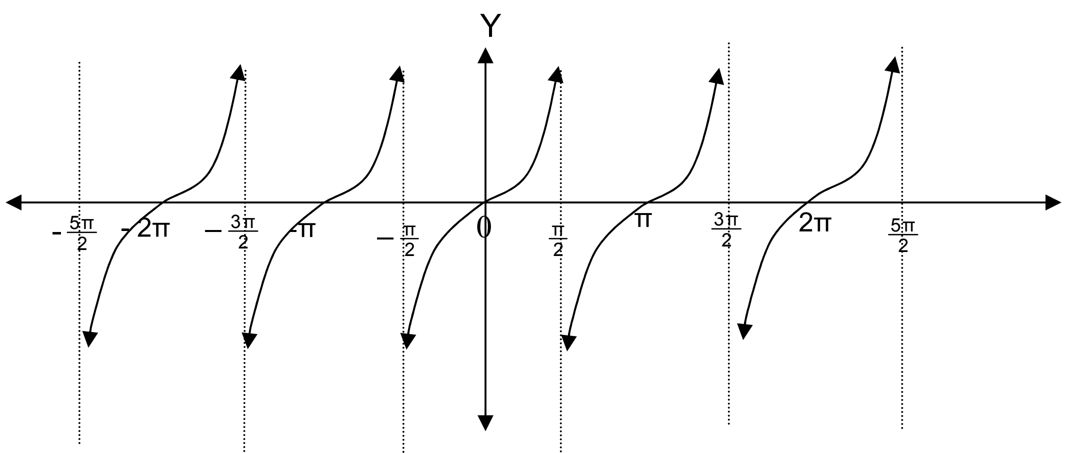
The graph of y = cos θ can be constructed in the same manner as the graph of y = sin θ, that is, by plotting the abscissa along the y-axis and the arc lengths along the x-axis. Observe the properties of the graph of y = cos θ for the interval -2π ≤ θ ≤ 2π shown below.



You will observe that just like the graph of y = sin θ, it is also a curve. It also has a period of 2π and amplitude 1. For the interval [0, 2π], the minimum point is (π, -1), maximum points are (0, 1) and (2π, 1) and the graph crosses the x-axis at $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$. The graph is decreasing from 0 to π and increasing from π to 2π over the interval [0, 2π].

The Graph of the Tangent Function

The graph of y = tan θ can be drawn in the same manner that the graphs of the sine and cosine functions. The value of the tangent of an angle is plotted along the y-axis and the arc lengths on the x-axis. Observe that the tangent of the odd multiples of $\frac{\pi}{2}(90^o)$ are not defined so that the graph is discontinuous at those values. These are denoted by the broken lines (called asymptotes) that separate one complete cycle from the others. Thus, the domain of these function exclude all odd multiples of $\frac{\pi}{2}$ while the range is the set of real numbers.



Notice that the period of the graph of the tangent function is π. This is shown by a complete curve in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. See that the other curves are repetitions of the curve for the given interval. The graph of the tangent function is also said to be an odd function and that the graph is symmetrical with respect to the origin.

You will also see from the graph that it is an increasing function for the different sets of intervals.

Try this out

A. Refer to the graph of the y = sin θ to answer the following.

1. What is the domain of the sine function?

2. What is its range?
3. Give the intercepts of y = sin θ for the interval [-2π, 0]
4. Determine the interval where the graph of y = sin θ is (a) increasing, (b) decreasing for the interval [-2π, 0].

B. Refer to the graph of y = cos θ to answer the following.

1. What is the domain of the cosine function?
2. What is its range?
3. Give the intercepts of y = cos θ for the interval [-2π, 0]
4. Determine the interval where the graph of y = cos θ is (a) increasing, (b) decreasing for the interval [-2π, 0].

C. Refer to the graph of y = tan θ

1. What is the domain of the graph of y = tan θ?
2. What is its range?
3. At what values of θ in the graph is tangent not defined?
4. Give the vertical asymptotes of the graph?
5. What are the zeros of y = tan θ?

Lesson 2

Properties of Sine and Cosine functions

The two properties of Sine and Cosine functions are amplitude and a period of a function. This can be determined from a given equations. The function in the form of y = a sin bx and y a cos bx, the amplitude is /**a**/ and the period is $\frac{2\pi}{b}$.

Examples:

Determine the amplitude and the period of the given function:

1. y = 3 sin 2x

Solution:

$a = 3; \quad b = 2$

a. amplitude is $= |a|$
$= |3|$
$= 3$

b. period is $P = \frac{2\pi}{b}$

$= \frac{2\pi}{2}$

$= \pi$

2. $y = \frac{1}{2}\cos x$

Solution:

$a = \frac{1}{2}; \quad b = 1$

a. amplitude is $= |a|$

$= \left|\frac{1}{2}\right|$

$= \frac{1}{2}$

b. period is $P = \frac{2\pi}{b}$

$= \frac{2\pi}{1}$

$= 2\pi$

3. $y = -3 \sin 4x$

Solution:

$a = -3; \quad b = 4$

a. amplitude is $= |a|$

$= /-3/$

$= 3$

b. period is P = $\frac{2\pi}{b}$

$= \frac{2\pi}{4}$

$= \frac{\pi}{2}$

4. y = cos 4x

Solution:

a = 1; b = 4

a. amplitude is = /a /

$= /1/$

$= 1$

b. period is P = $\frac{2\pi}{b}$

$= \frac{2\pi}{4}$

$= \frac{\pi}{2}$

5. y = -$\frac{2}{3}$ sin x

Solution:

a = -$\frac{2}{3}$; b = 1

a. amplitude is = /a/

$= /-\frac{2}{3}/$

$= \frac{2}{3}$

b. period is P = $\frac{2\pi}{b}$

$= \frac{2\pi}{1}$

$= 2\pi$

Try this out

A. Determine the amplitude of the following functions.

1. $y = 3 \sin x$
2. $y = 2 \sin \frac{1}{2}x$
3. $y = \frac{3}{2} \cos 2x$
4. $y = -2 \cos 2x$
5. $y = \cos 4x$
6. $y = 2 \sin 2x$
7. $y = \frac{1}{2} \sin x$
8. $y = \frac{3}{4} \sin \frac{1}{2}x$
9. $y = -4 \sin 3x$
10. $y = -2 \sin \frac{3}{2} x$

B. Determine the period of the following functions.

1. $y = 6 \sin \frac{2}{3}x$
2. $y = 5 \sin x$
3. $y = 4 \sin \frac{1}{2}x$

4. $y = -\frac{1}{2} \cos \frac{3}{4}x$

5. $y = 3 \cos \frac{1}{2}x$

6. $y = \cos 3x$

7. $y = -3 \sin \frac{2}{3}x$

8. $y = -6 \sin 2x$

9. $y = 2 \sin \frac{1}{5}x$

10. $y = \sin 4x$

Lesson 3

Trigonometric Equations

In this section we will solve trigonometric equations using your knowledge in solving algebraic equations. We will also find values which are true for the domain of the variables under some given conditions.

Examples:

1. Find θ in $\sqrt{3}\cos\theta - 2 = 0$ in the interval $0 \leq \theta \leq 2\pi$.

   Solution:

$$2\cos\theta - \sqrt{3} = 0$$

$$2\cos\theta = \sqrt{3}$$

$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6} \quad \text{the reference angle}$$

In the interval $0 \leq \theta \leq 2\pi$, $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$.

Since the cosine function has a period of 2π, we can obtain the general solution by adding multiple of 2π. We have,

$$\theta = \frac{\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi, where\ n\ is\ an\ \mathrm{int}eger.$$

or θ = 30° + 360°n, 330° + 360°n wnere n is an integer.

2. sec x = - $\sqrt{2}$ in the interval 0 < θ < 2π.

Solution:

sec x = - $\sqrt{2}$

Since sec x is $\frac{1}{\cos x}$, then

cos x = -$\frac{1}{\sqrt{2}}$

= $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

cos x = -$\frac{\sqrt{2}}{2}$

x = $\frac{\pi}{4}$ the reference angle

Since x is in the interval interval 0 < x < 2π and cos x is negative, then x = $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$.

3. Determine the solution set of $\tan^2 \theta$ = tan θ in the interval 0 < θ < 2π.

Solution:

$\tan^2 \theta$ = tan θ

$\tan^2 \theta$ - tan θ = 0

tan θ(tan θ - 1) = 0

tan θ = 0 tanθ - 1 = 0

tan θ = 1

θ = 0, π $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

The solution of the $\tan^2 \theta$ = tan is $\{0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}\}$

4. Determine the solution set of sin 2x = 0 in the interval $0 \leq x \leq 2\pi$.

Solution:

cos 2x = 0

Since $0 \leq x \leq 2\pi$ then $0 \leq 2x \leq 4\pi$.

.

Then $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

and $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The solution set of the given equation is $\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$.

Try this out

Determine the solution of the following equation in the interval $0 \leq x \leq 2\pi$.

1. tan x sin x = 0
2. $2 \cos^2 x + \cos x = 0$
3. $2 \sin^2 x + 5 \cos x – 3 = 0$
4. tan 2x = 1
5. $4 \sin^2 x = 3$

6. $\cot^2 x - 1 = 0$

7. $4\sin x \cos x = -\sqrt{3}$

8. $3 \cos x = -6$

9. $\cot^2 x - 1 = 0$

10. $(\cos x - 1)(\cos x + 1) = 0$

Let's Summarize

1. The graph of the sine function is periodic. The period is 2π. Its domain is the set of real number and range is [-1, 1]. Its amplitude is 1 and the curve crosses the x-axis at the odd multiples of $\frac{\pi}{2}$. It has a maximum value 1 and a minimum value -1. The graph is increasing in the interval $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$ while decreasing in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ over the period 2π.

2. The graph of the cosine function is periodic with a period 2π. Its domain is the set of real number and range is [-1, 1]. Its amplitude is 1 and the curve crosses the x-axis at the multiples of π. It has a maximum value 1 and a minimum value -1. The graph is increasing in the interval [π, 2π] while decreasing over the interval [0, π].

3. The graph of the tangent function is periodic with a period π. Its domain is the set of real numbers except the odd multiples of $\frac{\pi}{2}$ where tangent is undefined. The range is the set of real numbers. It is an odd function and has vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

4. The function in the form of y = a sin bx and y a cos bx, the amplitude is **/a/** and the period is $\frac{2\pi}{b}$.

What have you learned

Given the following function, identify the amplitude of:

1. y = $\frac{1}{2}$ cos x

2. y = 4 cos x

3. y = sin 4x

Determine the period of the following functions.

4. y = 3 sin x

5. y = cos x

6. y = 4 sin 4x

7. What is the value of y = $\frac{1}{2}$ cos x if x = $\frac{1}{2}$

8. What is the value of y = 3 tan 2x , it x = $\frac{7\pi}{6}$.

9. The period of the tangent function is

   a. 2π c. π

   b. $\frac{\pi}{2}$ d. $\frac{3\pi}{2}$

.

10. What is the amplitude of the sine function?

   a. 1   c. 2

   b. -1   d. -2

11. Which of the following are zeros of $y = \cos \Theta$ for the interval $0 \le \Theta \le 2\pi$?

a. 0, π and 2π

b. $\frac{\pi}{2}$ and $\frac{3\pi}{2}$

c. $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

d. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

12. which of the intervals is the sine function increasing over the period of 2π?

a. $[0, \pi]$

b. $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$

c. $[\pi, 2\pi]$

d. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

13. Determine the solution of $\sin \frac{1}{3}x = 0$ in the interval $0 \le \Theta \le 2\pi$.

Answer Key

How much do you know

1. a
2. d
3. y = 2
4. $\sqrt{2}$
5. 2
6. $\frac{3}{4}$
7. -2
8. 4π
9. 3π
10. 10π
11. a
12. a
13. $\frac{\pi}{2}$

Try this out

Lesson 1

A. 1. Real Numbers
   2. [-1, 1]
   3. 0, -π, -2π
   4. increasing: $\left[-2\pi, -\frac{3\pi}{2}\right]$, $\left[-\frac{\pi}{2}, 0\right]$

      decreasing: $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$

B. 1. Real Numbers
   2. [-1, 1]

3. $-\frac{\pi}{2}, -\frac{3\pi}{2}$

4. increasing: [π, 0]

   decreasing: [-2π, -π]

C. 1. Real numbers except the odd multiples of $\frac{\pi}{2}$

2. Real Numbers

3. all odd multiples of $\frac{\pi}{2}$; $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

4. all odd multiples of $\frac{\pi}{2}$; $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

5. -2π, - π, 0, π, 2 π

Lesson 2

A.

1. amplitude : 3
2. amplitude 2
3. amplitude: $\frac{3}{2}$
4. amplitude: 2
5 amplitude: 1
6. amplitude: 2
7. amplitude: $\frac{1}{2}$
8. amplitude: $\frac{3}{4}$
9. amplitude: 4
10. amplitude: 2

B.

1. Period: 3π
2. Period: 2π
3. Period 4π
4. Period: $\frac{8\pi}{3}$

5. Period : 4π
6. Period : $\frac{2\pi}{3}$
7 Period : 3π
8. Period: π
9. Period: 10π
10. Period: $\frac{\pi}{2}$

Lesson 3

1. tan x sin x = 0

   Solution:

   $\tan x \sin x = 0$

   $\tan x = 0$ $\qquad\qquad$ $\sin x = 0$

   $x = \pi, 2\pi,$ $\qquad\qquad$ $x = \pi, 2\pi$

   The solution set of tan x sin x = 0 is π and 2π.

2. $2\cos^2 x + \cos x = 0$

   Solution:

   $2\cos^2 x + \cos x = 0$

   $\cos x(2\cos x - 1) = 0$

   $\cos x = 0$ $\qquad\qquad$ $2\cos x - 1 = 0$

   $x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad\qquad$ $\cos x = \frac{1}{2}$

   $x = \frac{\pi}{3}, \frac{5\pi}{3}$

   The solution of $2\cos^2 x + \cos x = 0$ is $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\}$.

3. $2 \sin^2 x + 5 \sin x - 3 = 0$

Solution:

$2 \sin^2 x + 5 \sin x - 3 = 0$

$(2 \sin x - 1)(\sin x + 3) = 0$

$2 \sin x - 1 = 0$ $\qquad\qquad$ $\sin x + 3 = 0$

$2 \sin x = 1$ $\qquad\qquad$ $\sin x = -3$

$x =$ no solution

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

4. $\tan 2x = 1$

Solution:

$\tan 2x = 1$

$0 \leq 2x \leq 4\pi$ $\qquad\qquad$ $0 \leq x \leq 2\pi$

$2x = \frac{\pi}{4}, \frac{5\pi}{4}$ $\qquad\qquad$ $x = \frac{\pi}{8}, \frac{5\pi}{8}$

The solution set is $\frac{\pi}{8}, \frac{5\pi}{8}$.

5. $4 \sin^2 x = 3$

Solution:

$4 \sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}$

The solution set is $\{\frac{\pi}{3}, \frac{2\pi}{3}\}$.

6. $\cot^2 x - 1 = 0$

Solution:

$\cot^2 x - 1 = 0$

$(\cot x - 1)(\cot x + 1) = 0$

$\cot x - 1 = 0$ $\qquad$ $\cot x + 1 = 0$

$\cot x = 1$ $\qquad$ $\cot x = -1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ $\qquad$ $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

The solution set is $\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$

7. $2 \cos x - \sqrt{3} = 0$

Solution:

$2 \cos x - \sqrt{3} = 0$

$2 \cos x = \sqrt{3}$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

The solution is $\{\frac{\pi}{6}, \frac{11\pi}{6}\}$

8. $3 \cos x = -6$

Solution:

$3 \cos x = -6$

$\cos x = -2$

No solution, because all values of cos x are between 1 and -1.

9. $4 \cos^2 x = 1$

Solution:

$4 \cos^2 x = 1$

$\cos^2 x = \frac{1}{4}$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

The solution set is $\{\frac{\pi}{3}, \frac{5\pi}{3}\}$

10. $(\cos x - 1)(\cos x + 1) = 0$

Solution:

$(\cos x - 1)(\cos x + 1) = 0$

$\cos x - 1 = 0$ $\qquad$ $\cos x + 1 = 0$

$\cos x = 1$ $\qquad$ $\cos x = -1$

$x = 0, 2\pi$ $\qquad$ $x = \pi$

The solution set is $\{0, \pi, 2\pi\}$

What have you learned

1. $\frac{1}{2}$
2. 4
3. 1
4. 2π
5. 2π
6. $\frac{\pi}{2}$
7. 0
8. $3\sqrt{3}$
9. c
10. 1
11. a
12. b
13. $\{\frac{\pi}{3}, \frac{2\pi}{3}\}$