Module 2
Circular Functions and Trigonometry

What this module is about

This module is about determining the coordinates of angles in standard position in a unit circle; the six circular functions and finding the six circular functions of special angles; As you go over the discussion, examples and exercises, you will understand what circular functions are all about. Anytime you feel you are at a loss, do not hesitate to go back to the discussion and examples.

What you are expected to learn

This module is designed for you to:

1. determine the coordinates of the terminal side of an angle in standard position in a unit circle

   1.1 when one coordinate is given (apply the Pythagorean Theorem and the properties of special right triangles)

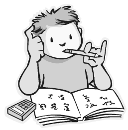
   1.2 when the angle is of the form:

   - $180^\circ n \pm 30^\circ$
   - $180^\circ n \pm 60^\circ$
   - $180^\circ n \pm 45^\circ$
   - $90^\circ n$

2. define the six circular functions

   - sine
   - cosine
   - tangent
   - cotangent
   - secant
   - cosecant

3. find the six circular functions of angles with special values



How much do you know

1. The x-coordinate of an angle in the along the unit circle is $\frac{3}{4}$. If the terminal side of the angle is located in the fourth quadrant, what is its y-coordinate?

   a. $\frac{1}{4}$ b. $\frac{\sqrt{7}}{4}$ c. $-\frac{\sqrt{7}}{4}$ d. $-\frac{1}{4}$

2. An angle measuring 30^o is in standard position along the unit circle. What are its coordinates?

   a. $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ b. $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ c. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ d. $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

3. What are the coordinates of the point of intersection of the terminal side of a 420^o angle and the unit circle?

   a. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ b. $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ c. $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ d. $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

4. What is the y-coordinate of a 225^o angle along the unit circle?

   a. $-\frac{1}{2}$ b. $-\frac{\sqrt{3}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. -1

5. It is the relationship between the arc length and the x-coordinate.

   a. Circular function
   b. Sine function
   c. Cosine function
   d. Tangent function

6. What is cos 120^o

   a. $\frac{1}{2}$ b. $-\frac{1}{2}$ c. $\frac{\sqrt{3}}{2}$ d. $-\frac{\sqrt{3}}{2}$

What you will do

Lesson 1

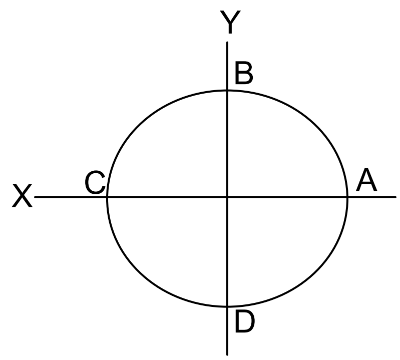
Coordinates of Points on the Unit Circle

In the previous module you have learned about the measures of arcs on a unit circle. Now, let us find the coordinates of the point where the terminal side of an angle in standard position lies.

Consider a circle whose center is at (0,0). The circle of radius one with center at origin is called the unit circle. Every point on the unit circle satisfies the equation $x^2 + y^2 = 1$.

It intersects at the points:

A: (1,0)
B: (0, 1)
C: (-1, 0)
D: (0, -1)



You can determine whether a point is on the unit circle if the equation $x^2 + y^2 = 1$ is satisfied.

Examples:

Determine whether each point lie on the unit circle.

a. $\left(\frac{12}{13}, \frac{-5}{13} \right)$ c. $\left(\frac{1}{3}, \frac{1}{2} \right)$

b. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$ d. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$

Solution:

a. $\left(\frac{12}{13}, \frac{-5}{13} \right)$ implies that $x = \frac{12}{13}$ and $y = \frac{-5}{13}$

Substitute the values of x and y in $x^2 + y^2 = 1$

$(\frac{12}{13})^2 + (\frac{-5}{13})^2 = 1$

$\frac{144}{169} + \frac{25}{169} = 1$

$\frac{169}{169} = 1$ True

The point $(\frac{12}{13}, \frac{-5}{13})$ is on the unit circle.

b. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ implies that $x = -\frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$

Substitute the values of x and y in $x^2 + y^2 = 1$

$(-\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2 = 1$

$\frac{2}{4} + \frac{2}{4} = 1$

$\frac{4}{4} = 1$ True

The point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ lie on the unit circle.

c. $(\frac{1}{3}, \frac{1}{2})$

$(\frac{1}{3}, \frac{1}{2})$ implies that $x = \frac{1}{3}$, and $y = \frac{1}{2}$

Substitute the values of x and y in $x^2 + y^2 = 1$

$(\frac{1}{3})^2 + (\frac{1}{2})^2 = 1$

$\frac{1}{9} + \frac{1}{4} = 1$

$$\frac{4+9}{36} = 1$$

$$\frac{13}{36} = 1 \quad \text{False}$$

The point ($\frac{1}{3}$, $\frac{1}{2}$) is not on the unit circle.

d. ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$)

($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$) implies that $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$

Substitute the values of x and y in $x^2 + y^2 = 1$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

$$\frac{4}{4} = 1 \quad \text{True}$$

The point ($\frac{1}{2}$, $\frac{\sqrt{3}}{2}$) is on the unit circle.

Knowing this equation, the other coordinate of a point of intersection of the unit circle and the terminal side of an angle in standard position can be obtained when one of its coordinates is given.

Example 1:

If the x-coordinate of an angle in standard position is $\frac{1}{2}$, what is the y-coordinate?

Solution:

Use the equation of the unit circle, $x^2 + y^2 = 1$. Substitute the given value of x in the equation of the unit circle to obtain the value of y.

$\left(\frac{1}{2}\right)^2 + y^2 = 1$ Substitute the given x-coordinate

$\frac{1}{4} + y^2 = 1$ $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$y^2 = 1 - \frac{1}{4}$ Addition property of equality

$y^2 = \frac{3}{4}$ $1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}$

$y = \pm\sqrt{\frac{3}{4}}$ Take the square root of both numerator and denominator

$y = \pm\frac{\sqrt{3}}{2}$ $\sqrt{4} = 2$

$\therefore y = \pm\frac{\sqrt{3}}{2}$.

Example 2:

The y-coordinate of an angle in standard position is $\frac{1}{7}$. If the terminal side of the angle lies between 90^o and 180^o, what is its x-coordinate?

Solution:

Use the equation of the unit circle $x^2 + y^2 = 1$. Substitute the given y-coordinate to find the x-coordinate.

$x^2 + \left(\frac{1}{7}\right)^2 = 1$ Substitute the given y-coordinate

$x^2 + \frac{1}{49} = 1$ $\left(\frac{1}{7}\right)^2 = \frac{1}{49}$

$x^2 = 1 - \frac{1}{49}$ Addition Property of Equality

$x^2 = \frac{48}{49}$ $1 - \frac{1}{49} = \frac{49-1}{49} = \frac{48}{49}$

$$x = \pm\sqrt{\frac{48}{49}}$$ Take the square root of both sides

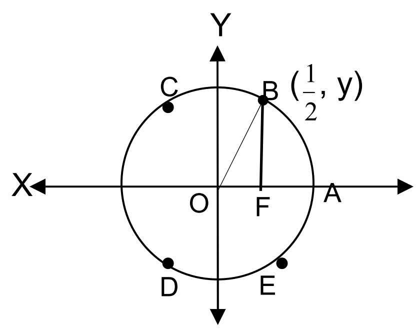
$$x = \pm\sqrt{\frac{16 \cdot 3}{49}}$$

$$x = \pm\frac{4\sqrt{3}}{7}$$

$\therefore x = -\frac{4\sqrt{3}}{7}$, since $90^o < \theta < 180^o$ which means that the terminal side of the angle lies in the second quadrant where the x-coordinate is negative.

Try this out

A. Let B be a point on the first quadrant of the unit circle. The x-coordinate of the point of intersection of the unit circle and the terminal side at B is $\frac{1}{2}$.



1. Name the angle in standard position.
2. What special kind of triangle is formed out of points O, B, and F?
3. How will you find the y-coordinate of B?
4. Find the y-coordinate of $\angle$AOB.
5. What is the sign of the y-coordinate of B?
6. What then are the coordinates of B?
7. What are the coordinates of point C which is a reflection of point B in the second quadrant?
8. If D is a reflection of C in the third quadrant, what are the coordinates of the terminal point of $\angle$AOD?

9. If E is a reflection of B in the fourth quadrant, what are the coordinates of the terminal point of $\angle AOE$?

10. In what quadrant is the abscissa or x-coordinate of the terminal point of an angle positive? Negative?

11. What about the y-coordinate or ordinate? In what quadrant is it positive? Negative?

B. Determine whether each of the following points lie on the unit circle.

1. $\left(\frac{3}{5}, \frac{4}{5} \right)$
2. $(0.8, -0.6)$
3. $\left(2\frac{2}{3}, \frac{-1}{3} \right)$
4. $(2 , -1)$
5. $\left(\frac{-8}{17}, \frac{-15}{17} \right)$

C. One of the coordinates of the point of intersection of the unit circle and the terminal side of an angle in standard position is given. Find the other coordinate.

1. $x = -\frac{1}{2}$
2. $y = -\frac{2}{5}$
3. $y = \frac{1}{10}$
4. $x = \frac{1}{3}$
5. $x = \frac{\sqrt{2}}{2}$
6. $y = \frac{1}{2}$, $0 < \theta < 90^\circ$
7. $y = \frac{1}{7}$, $90^\circ < \theta < 180^\circ$
8. $x = -\frac{\sqrt{2}}{3}$, $180^\circ < \theta < 270^\circ$
9. $y = -\frac{1}{5}$, $270^\circ < \theta < 360^\circ$
10. $x = \frac{5}{7}$, $0 < \theta < 90^\circ$

D. Find the missing coordinate of the point of intersection of the unit circle and the terminal ray of an angle in standard position.

1. $\left(\frac{\sqrt{3}}{2}, \_\_\right)$ θ is in QI
2. $(\_\_, -0.6)$ θ is in QIII
3. $\left(\_\_, \frac{5}{13}\right)$ θ is in QII
4. $(\_\_, 1)$ $\theta = 90°$
5. $(0, \_\_)$ $\theta = 270°$
6. $\left(-\frac{12}{13}, \_\_\right)$ θ is in QII
7. $\left(\_\_, -\frac{3}{4}\right)$ θ is in QIII
8. $\left(\_\_, -\frac{\sqrt{3}}{5}\right)$ $180 < \theta < 270°$
9. $\left(\frac{2}{7}, \_\_\right)$ $0° < \theta < 90°$
10. $\left(\_\_, \frac{5}{12}\right)$ $90° < \theta < 180°$

Lesson 2

The Coordinates of Points on the Unit Circle in the form $180^o n \pm A$

In geometry, you have learned that angles whose measures are 30^o, 45^o, and 60^o are called special angles. Now, let these special angles be amount of rotations on the unit circle as you can see in the figures below. Angle measures 30, 45 and 60 are also the arclengths $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively

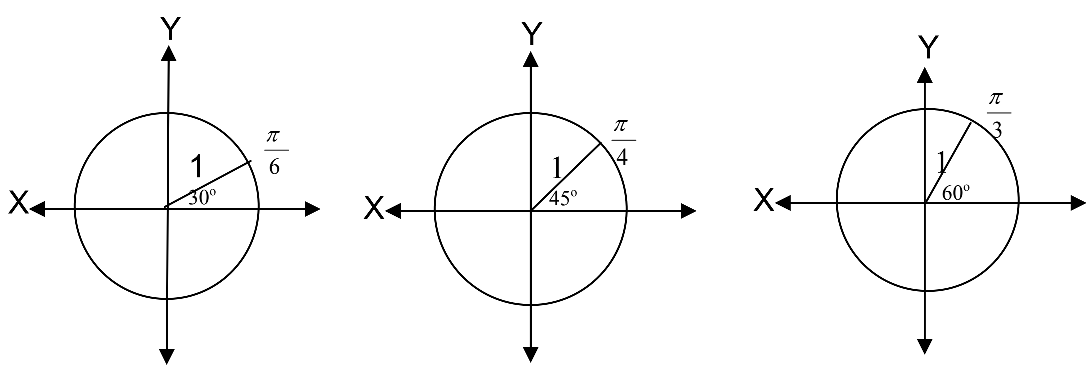


Figure 1 Figure 2 Figure 3

What do you think are the coordinates of each of the terminal points?

Consider figure 1. Let P(x, y) be the terminal point of the angle. Drop a perpendicular from P to the x-axis and call the point of intersection M. Notice that triangle OPM is a 30^o-60^o-90^o triangle.

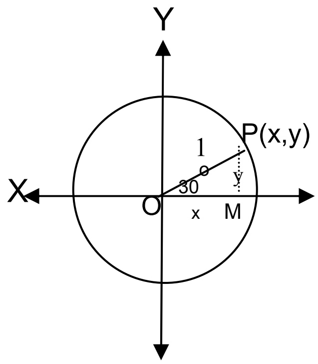


Figure 1

Recall that in a 30^o-60^o-90^o triangle, the length of the leg opposite the 30^o angle is $\frac{1}{2}$ the length of the hypotenuse. Hence, y = $\frac{1}{2}$.

The x-coordinate of P can be obtained using the equation of the unit circle $x^2 + y^2 = 1$.

By substitution,

$x^2 + \left(\frac{1}{2}\right)^2 = 1$ $\qquad x^2 = \frac{3}{4}$

$x^2 + \frac{1}{4} = 1$ $\qquad x = \pm\sqrt{\frac{3}{4}}$

$x^2 = 1 - \frac{1}{4}$ $\qquad x = \pm\frac{\sqrt{3}}{2}$

Since P is in QI, then x = $\frac{\sqrt{3}}{2}$. Thus, P ($\frac{\pi}{6}$)has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Consider this time figure 2. Let P(x, y) be the terminal point of the angle. Drop a perpendicular from P to the x-axis and call the point of intersection M. Notice that triangle OPM is a 45^o-45^o-90^o triangle and that a 45^o angle in standard position has its terminal side in the first quadrant.

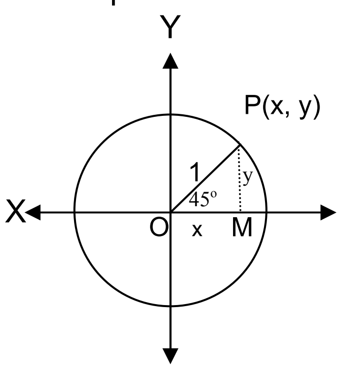


Figure 2

Recall that in a 45^o-45^o-90^o, the lengths of the two legs are equal. Thus, in Figure 2, x = y.

The coordinates of P can be obtained using the equation of the unit circle, $x^2 + y^2 = 1$ where x = y.

$x^2 + x^2 = 1$

$2x^2 = 1$ $\qquad x = \pm\sqrt{\frac{1}{2}}$

$x^2 = \frac{1}{2}$ $\qquad x = \pm\frac{1}{\sqrt{2}} \, or \pm \frac{\sqrt{\frac{\sqrt{2}}{2}}2}{2}$

But P is in quadrant I. Hence, $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$.

$\therefore$ P ($\frac{\pi}{4}$) has coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$.

Now, consider figure 3. Let P(x, y) be the terminal point of the angle. Drop a perpendicular from P to the x-axis and call the point of intersection M. Notice that ΔPOM is a right triangle where $\angle$POM = 60^o.

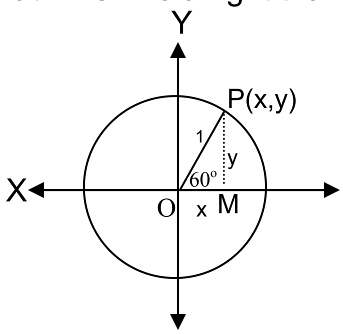


Figure 3

To determine the coordinates of P, use the same relation as in Figure 1 since $\angle$OPM = 30^o. Knowing that the side opposite 30^o is $\frac{1}{2}$ the length of the hypotenuse, then in figure 3, x = $\frac{1}{2}$.

To find y, use the equation of the unit circle, $x^2 + y^2 = 1$. Thus,

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

$$y^2 = 1 - \frac{1}{4}$$

$$y = \frac{3}{4}$$

$$y = \pm\sqrt{\frac{3}{4}}$$

$$y = \pm\frac{\sqrt{3}}{2}$$

Since P is in the first quadrant y = $\frac{\sqrt{3}}{2}$.

Hence, P($\frac{\pi}{3}$)has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$.

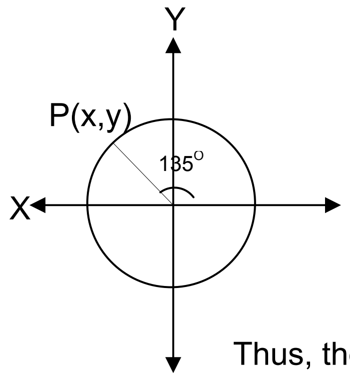
The coordinates of the terminal points of special angles on a unit circle can now be used to determine the coordinates of points of intersection of angles in the form $180^o n \pm A$ where A is a special angle. This can be done by determining the location of the terminal point of the given angle and the reference angle, A.

Example:

1. Determine the coordinates of the point of intersection of the terminal side of a 135^o angle on the unit circle.

Solution:

The terminal point of 135^o or $\frac{3\pi}{4}$ is located in the second quadrant. Its reference angle can be determined using the form $180^o n - A$.



$$135 = 180^o(1) - A$$
$$135^o = 180^o - A$$
$$A = 180^o - 135^o$$
$$A = 45^o$$

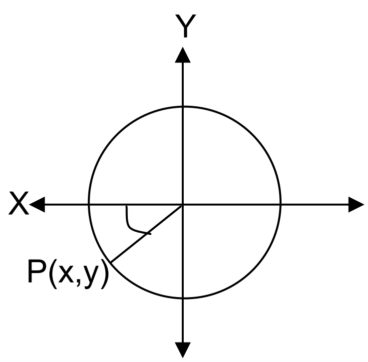
$\frac{3\pi}{4}$ or 135^o is a reflection of $\frac{\pi}{4}$ or 45^o in QII.

Thus, the terminal point P of 135^o has coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

2. Determine the coordinates of the point of intersection of the terminal side of a 210^o angle on the unit circle.

Solution:

The terminal point of 210^o or $\frac{7\pi}{6}$ is located in QIII. Its reference angle can be determined using the form $180^o n + A$.



$$180^o(1) + A = 210^o$$
$$180^o + A = 210^o$$
$$A = 210^o - 180^o$$
$$A = 30^o$$

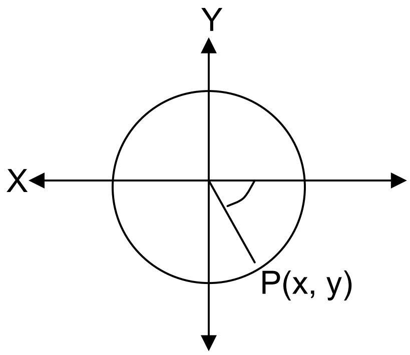
Notice that 210^o is a reflection of 30^o in the third quadrant. Hence, its coordinates can be obtained using the coordinates of 30^o but following the signs of the ordered pairs in the third quadrant.

Hence, the terminal point P of 210° or $\frac{7\pi}{6}$ has coordinates $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

3. Determine the coordinates of the point of intersection of the terminal side of a 660° angle on the unit circle.

Solution:

The terminal side of 660° or $\frac{11\pi}{3}$ is in the fourth quadrant. It can be written in the form 180°n - A.



$$180^\circ(4) - A = 660^\circ$$
$$-A = 660^\circ - 720^\circ$$
$$-A = -60^\circ$$
$$A = 60^\circ$$

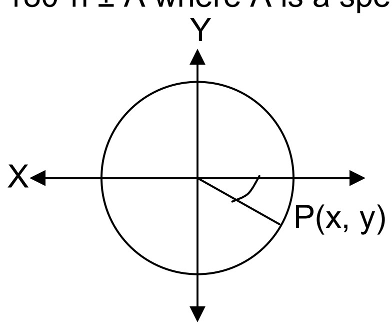
Observe that 660° is a reflection of 60° on the fourth quadrant. Hence, the coordinates of the terminal point can be obtained from the coordinates of 60° but following the signs of coordinates in the fourth quadrant.

Then, P($\frac{11\pi}{3}$) has coordinates $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

4. Determine the coordinates of the point of intersection of the terminal side of a -750° angle on the unit circle.

Solution:

The direction of rotation of -750° or $-\frac{25\pi}{6}$ is clockwise starting from (1, 0). Its terminal side is located in the fourth quadrant and be expressed in the form 180°n ± A where A is a special angle.



$$180^\circ(-4) - A = -750^\circ$$
$$-720^\circ - A = -750^\circ$$
$$-A = -750 + 720^\circ$$
$$-A = -30^\circ$$
$$A = 30^\circ$$

Thus, the reference angle of -750° is 30°. The coordinates of the terminal point can be obtained from the coordinates of the points of intersection of the terminal side of 30° and the unit circle.

Therefore, the coordinates of the point of intersection of the terminal side of -750° or $-\frac{25\pi}{6}$ and the unit circle are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Observe from the examples that the coordinates of the point of intersection of the unit circle and the terminal side of an angle in standard position are as follows:

1. If an angle is of the form $180°n \pm 30°$, the coordinates are $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$.
2. If an angle is of the form $180°n \pm 60°$, the coordinates are $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.
3. If an angle is of the form $180°n \pm 45°$, the coordinates are $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

The signs of the coordinates depend upon the position of the terminal side of the angle.

Try this out

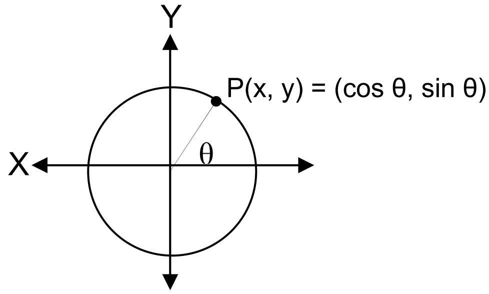
Find the reference angle for each of the following angles on a unit circle and determine the coordinates of its terminal point.

1. 120°	6. 315°	11. 765°	16. -225°
2. 150°	7. 330°	12. -120°	17. -300°
3. 225°	8. 480°	13. -135°	18. -480°
4. 240°	9. 510°	14. -150°	19. -600°
5. 300°	10. 585°	15. -210°	20. -1020°

Lesson 3

The Sine and Cosine Functions

In the previous lesson, you have learned that each terminal point P of special angles on the unit circle corresponds to coordinates of a point (x, y) that satisfies the equation of the unit circle $x^2 + y^2 = 1$. These coordinates of points have special names. The relation between the angle and the y-coordinate is called the **sine function** while the relation between the angle and the x-coordinate is called the **cosine function**.



In the figure at the left, the terminal side of angle θ intersected the unit circle at point P(x, y). Thus, the x-coordinate of P is called the cosine function of θ and can be expressed as cos θ = x, and the y-coordinate of P is called the sine function of θ and can be expressed as sin θ = y.

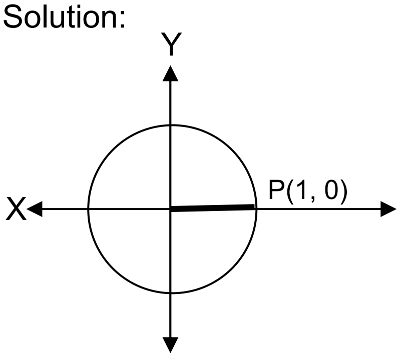
Hence, for each value of θ in the unit circle, the x-coordinate of the terminal point is the cosine of θ and the y-coordinate is the sine of θ.

In symbols, $x = \cos\theta$ and $y = \sin\theta$

Examples:

1. Evaluate $\sin 0^o$ and $\cos 0^o$.

Solution:

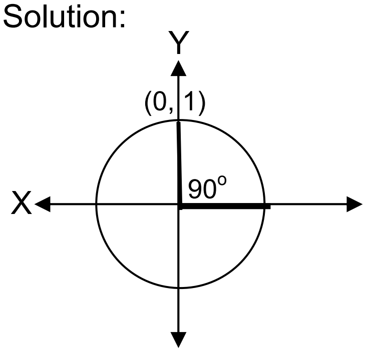


When $\theta = 0^o$ is set in standard position, it intersects the unit circle at the point (1, 0). By definition, $\sin\theta = y$ and $\cos\theta = x$.

Hence, in P(1,0) where x = 1 and y = 0
$\sin 0^o = 0$ and $\cos 0^o = 1$.

2. Evaluate $\sin 90^o$ and $\cos 90^o$. (Note: $90^o = \frac{\pi}{2}$)

Solution:



$\theta = 90^o$ is set in standard position on the unit circle as shown in the figure. Its terminal side intersects the unit circle at the point (0, 1). By definition, $\sin\theta = y$ and $\cos\theta = x$.

Hence, $\sin 90^o = 1$ and $\cos 90^o = 0$ or

$$\sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0.$$

Note that the sine and cosine functions of angles which are integral multiples of 90^o can be easily evaluated.

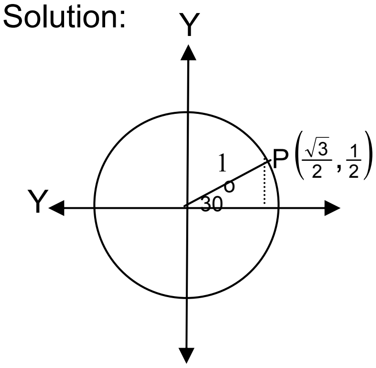
The table below shows the sine and cosine functions of the integral multiples of 90^o for $0^o \le \theta \le 360^o$.

θ	0	90^o or $\frac{\pi}{2}$	180^o or π	270^o or $\frac{3\pi}{2}$	360^o or 2π
P(θ)	(1, 0)	(0, 1)	(-1, 0)	(0, -1)	(1, 0)
sin θ	0	1	0	-1	0
cos θ	1	0	-1	0	1

To evaluate angles that are not multiples of 90^o, use the concept of reference angles as in the previous lesson.

3. Evaluate sin 30^o and cos 30^o. (Note: $30^o = \frac{\pi}{6}$)

Solution:



$\theta = 30^o$ is set in standard position, as in the figure at the left, the terminal side of the angle intersects the unit circle at the point P$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

By definition,

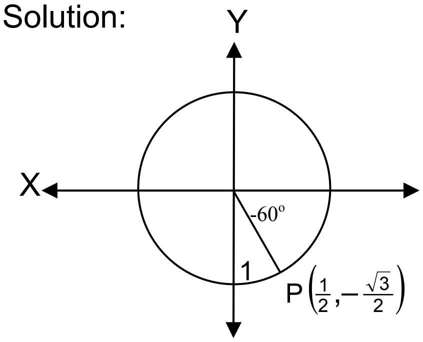
$\sin 30^o = \frac{1}{2}$ and $\cos 30^o = \frac{\sqrt{3}}{2}$ or

$\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Similarly, angles of multiples of 30^o or $\frac{\pi}{6}$ can be found in the other quadrants: 150^o or $\frac{5\pi}{6}$ in QII, 210^o or $\frac{7\pi}{6}$ in QIII and 330^o or $\frac{11\pi}{6}$ in QIV.

4. Evaluate sin -60^o and cos -60^o. (Note: 60^o = $\frac{\pi}{3}$

Solution:



If θ = -60^o or -$\frac{\pi}{3}$ is in standard position as in the figure, its terminal side intersects the unit circle at the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

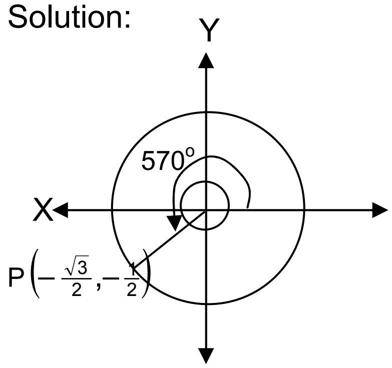
Hence, by definition,

$$\sin -60^o = \frac{1}{2} \text{ and } \cos -60^o = -\frac{\sqrt{3}}{2} \text{ or}$$

$$\sin -\frac{\pi}{3} = \frac{1}{2} \text{ and } \cos -60^o = -\frac{\sqrt{3}}{2}.$$

5. Evaluate sin 570^o and cos 570^o. (Note: 570^o = $\frac{19\pi}{6}$)

Solution:



The figure at the left shows 570^o in standard position. Its reference angle is 30^o and the terminal side intersects the unit circle at the point $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

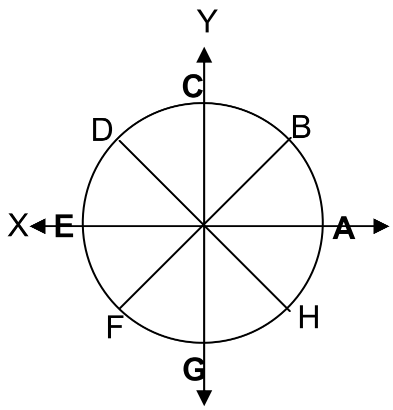
Hence, by the definition of sine and cosine functions,

$$\sin 570^o = -\frac{1}{2} \text{ and } \cos 570^o = -\frac{\sqrt{3}}{2} \text{ or}$$

$$\sin \frac{19\pi}{6} = -\frac{1}{2} \text{ and } \cos \frac{19\pi}{6} = -\frac{\sqrt{3}}{2}.$$

Try this out

A. The unit circle below is divided into 8 congruent arcs. Complete the table.



Terminal Point	Degree Measure	Coordinates	Sin θ	Cos θ
A				
B				
C				
D				
E				
F				
G				
H				

B. Evaluate the following:

1. $\sin 60^o$
2. $\cos 120^o$
3. $\cos 135^o$
4. $\sin 150^o$
5. $\cos 270^o$
6. $\sin 420^o$
7. $\sin -45^o$
8. $\cos -90^o$
9. $\sin -180^o$
10. $\cos -330^o$

C. Identify the quadrant/quadrants where the angle is/are located:

1. $\sin \theta > 0$
2. $\cos \theta < 0$
3. $\cos \theta > 0$
4. $\sin \theta < 0$
5. $\sin \theta > 0$ and $\cos \theta < 0$
6. $\sin \theta > 0$ and $\cos \theta > 0$
7. $\sin \theta < 0$ and $\cos \theta > 0$
8. $\sin \theta < 0$ and $\cos \theta < 0$

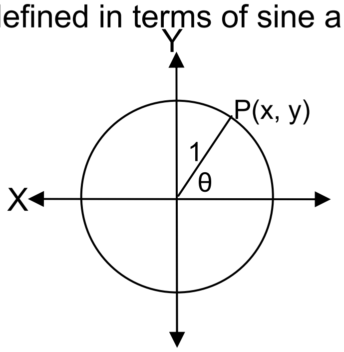
Lesson 4

The Other Circular Functions

In Lesson 3, you learned about two circular functions of an angle θ, sine and cosine. Aside from these two functions, there are four other circular functions of an angle θ in standard position. These are the tangent function, cotangent function, secant function and cosecant function.

The Tangent Function

The third basic function is the tangent function (abbreviated as **tan**). This function is defined in terms of sine and cosine functions.



The tangent function is defined as the set of all ordered pairs $\left(\theta, \frac{y}{x}\right)$ where x ≠ 0, θ is an angle in standard position and y and x are the second and first coordinates of the point of intersection of the terminal side of θ with the unit circle, respectively.

Since y = sin θ and x = cos θ, then tan θ = $\frac{\sin\theta}{\cos\theta}$ or $\frac{y}{x}$, where cos θ ≠ 0.

The Cotangent Function

The cotangent function (abbreviated as **cot**) is defined as the set of all ordered pairs $\left(\theta, \frac{x}{y}\right)$ where y ≠ 0, θ is an angle in standard position and x and y are the first and second coordinates of the point of intersection of the terminal side of θ with the unit circle, respectively.

Since x = cos θ and y = sin θ, then cot θ = $\frac{\cos\theta}{\sin\theta}$ or $\frac{x}{y}$, where sin θ ≠ 0.

The Secant Function

The secant function(abbreviated as **sec**) is defined as the set of all ordered pairs $\left(\theta, \frac{1}{x}\right)$ where x ≠ 0, θ is an angle in standard position and x = cos θ.

Since x = cos θ, then sec θ = $\frac{1}{\cos\theta}$ or $\frac{1}{x}$, where cos θ ≠ 0.

The Cosecant Function

The cosecant function (abbreviated as **csc**)is defined as the set of all ordered pairs $\left(\theta, \frac{1}{y}\right)$ where $y \neq 0$, θ is an angle in standard position and $y = \sin\theta$.

Since $y = \sin\theta$, then $\csc\theta = \frac{1}{\sin\theta}$ or $\frac{1}{y}$, where $\sin\theta \neq 0$.

Did you notice that tangent and cotangent functions are reciprocal functions? The same is true for secant and cosecant functions.

The value of the tangent, cotangent, secant and cosecant of special angles can be obtained using their x and y coordinates.

Examples:

1. Evaluate tan 30°, cot 30°, sec 30° and csc 30°. (Note: $30^\circ = \frac{\pi}{6}$)

Solution:

$P(30^\circ) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Hence, $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$.

Therefore, by definition of tangent, cotangent, secant and cosecant,

$$\tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \bullet \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec 30^\circ = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \bullet \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \bullet \frac{2}{1} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\csc 30^\circ = \frac{1}{\frac{1}{2}} = 1 \bullet \frac{2}{1} = \frac{2}{1} = 2$$

2. Determine tan 45°, cot 45°, sec 45° and csc 45°. (Note: $45^\circ = \frac{\pi}{4}$)

Solution:

If $P(45^\circ) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Hence, $\cos 45^\circ = \frac{\sqrt{2}}{2}$ and $\sin 45^\circ = \frac{\sqrt{2}}{2}$.

Therefore, by definition of tangent, cotangent, secant and cosecant,

$$\tan 45^o = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \bullet \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

$$\sec 45^o = \frac{1}{\frac{\sqrt{2}}{2}} = 1 \bullet \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\cot 45^o = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \bullet \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

$$\csc 45^o = \frac{1}{\frac{\sqrt{2}}{2}} = 1 \bullet \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Notice that the tan 45^o and cot 45^o are equal and that sec 45^o and csc 45^o are also equal. It is because the right triangle formed by a 45^o-45^o-90^o is an isosceles right triangle.

3. Find tan 60^o, cot 60^o, sec 60^o and csc 60^o. (Note: $60^o = \frac{\pi}{3}$)

If $P(60^o) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Hence, $\cos 60^o = \frac{1}{2}$ and $\sin 60^o = \frac{\sqrt{3}}{2}$.

Therefore, by the definition of tangent, cotangent, secant and cosecant,

$$\tan 60^o = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \bullet \frac{2}{1} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\sec 60^o = \frac{1}{\frac{1}{2}} = 1 \bullet \frac{2}{1} = \frac{2}{1} = 2$$

$$\csc 60^o = \frac{1}{\frac{\sqrt{3}}{2}} = 1 \bullet \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 60^o = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \bullet \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Observe that tan 60^o = cot 30^o and sec 60^o = csc 30^o. Why is it so? It is because 30^o and 60^o are complementary angles. Hence, their functions are also complementary.

The tangent, cotangent, secant and cosecant functions of other angles in the form $180^o n \pm 30^o$, $180^o n \pm 45^o$ and $180^o n \pm 60^o$ can be obtained from the functions of 30^o, 45^o, and 60^o, respectively.

4. Evaluate tan 0^o, cot 0^o, sec 0^o, and csc 0^o.

Solution:

P(0^o) = (1, 0). Hence cos 0^o = 1 and sin 0^o = 0.

Therefore, by definition,

tan 0^o = $\frac{0}{1} = 0$ sec 0^o = $\frac{1}{1} = 1$

cot 0^o = $\frac{1}{0} = \text{undefined}$ csc 0^o = $\frac{1}{0} = \text{undefined}$

Division by zero is not defined. Hence, cot 0^o and csc 0^o are not defined.

5. Evaluate tan 90^o. (Note 90^o = $\frac{\pi}{2}$)

Solution:

If P(90^o) = (0, 1). Hence, cos 90^o = 0 and sin 90^o = 1.

Therefore, by definition,

tan 90^o = $\frac{1}{0} = \text{undefined}$ sec 90^o = $\frac{1}{0} = \text{undefined}$

cot 90^o = $\frac{0}{1} = 0$ csc 90^o = $\frac{0}{1} = 0$

Division by zero is not defined. Therefore, tan 90^o and sec 90^o are not defined.

The Circular Functions of other Angles

The circular functions of angles which are not multiples of the quadrantal angles and special angles can be obtained using a scientific calculator or a table of trigonometric functions. In this module, the use of a scientific calculator is encouraged for you to make use of the technology. Caution is given that before you use a scientific calculator, that is, you have to familiarize yourself with the model of the scientific calculator you are going to use. This module will not prescribe a particular scientific calculator.

Examples:

Use a scientific calculator to evaluate the following

1. $\sin 15^o$
2. $\cos 34^o15'$
3. $\cot 100^o$
4. $\tan 125^o40'$
5. $\sec 20^o05'$
6. $\csc 320^o$

Answers:

1. $\sin 15^o = 0.258819$
2. $\cos 34^o15' = 0.826590$
3. $\cot 100^o = -0.176327$
4. $\tan 125^o40' = -1.393357$
5. $\sec 20^o05' = 1.064743$
6. $\csc 320^o = -1.555724$

Note that the symbol ' means minutes. Thus, $34^o15'$ means 34 degrees 15 minutes. Answers in the examples are given up to six decimal places.

Try this out

A. Find the six circular functions of the following angles using the definition of the functions.

1. 120^o
2. 135^o
3. 150^o
4. 180^o
5. 210^o
6. 225^o
7. 240^o
8. 270^o
9. 300^o
10. 330^o

B. Find the value of the following using a scientific calculator or a trigonometric table.

1. $\sin 23^o$
2. $\cos 34^o$
3. $\tan 16^o$
4. $\cot 43^o$
5. $\sec 95^o$
6. $\csc 102^o$
7. $\tan 44^o23'$
8. $\cos 48^o16'$
9. $\sin 55^o20'$
10. $\cot 29^o29'$
11. $\sec 120^o30'$
12. $\cot 87^o50'$
13. $\cos 95^o15'$
14. $\tan 112^o47'$
15. $\csc 50^o10'$
16. $\cos 37^o25'$
17. $\sin 200^o52'$
18. $\cot 312^o45'$
19. $\tan 300^o35'$
20. $\sec 320^o28'$

C. Identify the quadrant/quadrants where the angle is located given the following conditions:

1. $\tan \theta > 0$
2. $\cot \theta < 0$
3. $\tan \theta > 0$ and $\sin \theta < 0$
4. $\cot \theta < 0$ and $\cos \theta > 0$
5. $\sec \theta > 0$ and $\sin \theta > 0$
6. $\cot \theta > 0$ and $\sin \theta < 0$

Let's Summarize

1. If $\overline{OA}$ (1, 0) is the initial side of an angle on the unit circle and P is any point on the unit circle then, $\overline{OP}$ is the terminal side of $\angle AOP$ and the coordinates of point P satisfy the equation of the unit circle, $x^2 + y^2 = 1$.

2. The coordinates of an angle in the form $180°n \pm 30°$ are $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$.

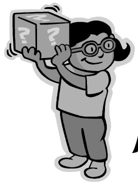
3. The coordinates of an angle in the form $180°n \pm 60°$ are $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

4. The coordinates of an angle in the form $180°n \pm 45°$ are $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

5. Sine function is the relation between an angle and the y-coordinate while cosine function is the relation between an angle and the x-coordinate.

6. Tangent function is the ratio of y to x while cotangent function is the ratio of x to y, where x and y are the coordinates of the point of intersection of the terminal side of an angle in standard position and the unit circle.

7. Secant function is the reciprocal of the cosine function while cosecant function is the reciprocal of the sine function.



What have you learned

1. The y-coordinate of an angle in the along the unit circle is $-\frac{4}{5}$. If the terminal side of the angle is located in the third quadrant, what is its x-coordinate?

   a. $\frac{3}{5}$ b. $\frac{1}{5}$ c. $-\frac{1}{5}$ d. $-\frac{3}{5}$

2. An angle measuring 60^o is in standard position along the unit circle. What are its coordinates?

   a. $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ c. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

   b. $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ d. $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

3. What are the coordinates of the point of intersection of the terminal side of a 600^o angle and the unit circle?

   a. $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ c. $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

   b. $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ d. $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

4. What is the x-coordinate of a 540^o angle along the unit circle?

   a. $-\frac{1}{2}$ b. $-\frac{\sqrt{3}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. -1

5. It is the reciprocal of the cosine function.

   a. Tangent function c. Cosecant function

   b. Sine Function d. Secant function

6. What is sec 240^o

   a. 2 b. $\frac{2\sqrt{3}}{3}$ c. -2 d. $-\frac{2\sqrt{3}}{3}$

Answer Key

How much do you know

1.	c	3.	b	5.	c	7.	a	9.	a
2.	c	4.	c	6.	b	8.	d	10.	a

Try this out
Lesson 1

A.
1. $\angle BOA$
2. Right triangle
3. Pythagorean Theorem
4. $\frac{\sqrt{3}}{2}$
5. positive
6. $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
7. $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
8. $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
9. $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
10. Pos: QI & QIV; Neg: QII & QIII
11. Pos: QI & QII; Neg: QIII & QIV

B.
1. Lies on the unit circle
2. Lies on the unit circle
3. Does not lie on the unit circle
4. Does not lie on the unit circle
5. Lies on the unit circle

C
1. $y = \pm\frac{\sqrt{3}}{2}$
2. $x = \pm\frac{\sqrt{21}}{5}$
3. $x = \pm\frac{3\sqrt{11}}{10}$
4. $y = \pm\frac{2\sqrt{2}}{3}$
5. $y = \pm\frac{\sqrt{2}}{2}$
6. $x = \frac{\sqrt{3}}{2}$
7. $x = -\frac{4\sqrt{3}}{7}$
8. $y = -\frac{\sqrt{7}}{3}$
9. $x = \frac{2\sqrt{6}}{5}$
10. $y = \frac{2\sqrt{6}}{7}$

D.
1. $\frac{1}{2}$
2. -0.8
3. $-\frac{12}{13}$
4. 0
5. -1
6. $\frac{5}{13}$
7. $-\frac{\sqrt{7}}{4}$
8. $-\frac{\sqrt{22}}{5}$
9. $\frac{3\sqrt{5}}{7}$
10. $-\frac{\sqrt{119}}{12}$

Lesson 2

1. 60^o $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
2. 30^o $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
3. 45^o $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
4. 60^o $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
5. 60^o $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
6. 45^o $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
7. 30^o $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
8. 60^o $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
9. 30^o $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
10. 45^o $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
11. 45^o $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
12. 60^o $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
13. 45^o $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
14. 30^o $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
15. 30^o $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
16. 45^o $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
17. 60^o $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
18. 60^o $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
19. 60^o $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
20. 60^o $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Lesson 3

A

Terminal Point	Degree Measure (θ)	Coordinates of Points	Sin θ	Cos θ
A	0^o	$(1, 0)$	0	1
B	45^o	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
C	90^o	(0, 1)	1	0
D	135^o	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
E	180^o	(-1, 0)	0	-1
F	225^o	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
G	270^o	(0, -1)	-1	0
H	315^o	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

B.
1. $\frac{\sqrt{3}}{2}$
2. $-\frac{1}{2}$
3. $-\frac{\sqrt{2}}{2}$
4. $\frac{1}{2}$
5. 0
6. $\frac{\sqrt{3}}{2}$
7. $-\frac{\sqrt{2}}{2}$
8. 0
9. 0
10. $\frac{\sqrt{3}}{2}$

C.
1. I, II
2. II, III
3. I, IV
4. III, IV
5. II
6. I
7. IV
8. III

Lesson 4

A.

Angle	Coordinates	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
1. 120^o	$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
2. 135^o	$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
3. 150^o	$\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
4. 180^o	(-1, 0)	0	-1	undefined	0	-1	undefined
5. 210^o	$\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2
6. 225^o	$\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
7. 240^o	$\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$
8. 270^o	(0, -1)	-1	0	undefined	0	undefined	-1
9. 300^o	$\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
10. 330^o	$\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2

B.
1. 0.390731
2. 0.829038
3. 0.286745
4. 1.072369
5. -11.47371
6. 1.022341
7. 0.978703
8. 0.665665
9. 0.822475
10. 1.768694
11. -1.979294
12. 0.037834
13. -0.091502
14. -2.380844
15. 1.302234
16. 0.794238
17. -0.356194
18. -0.924390
19. -1.692031
20. 1.296589

C. 1. I, III 2. II, IV 3. III 4. IV 5. I 6. III

What have you learned

1. d
2. a
3. b
4. d
5. d
6. c
7. c
8. a
9. b