Module 1
Circular Functions and Trigonometry

What this module is about

This module is about the unit circle. From this module you will learn the trigonometric definition of an angle, angle measurement, converting degree measure to radian and vice versa. The lessons were presented in a very simple way so it will be easy for you to understand and be able to solve problems alone without difficulty. Treat the lesson with fun and take time to go back if you think you are at a loss.

What you are expected to learn

This module is designed for you to:

1. define a unit circle, arc length, coterminal and reference angles.
2. convert degree measure to radian and radian to degree.
3. visualize rotations along the unit circle and relate these to angle measures.
4. illustrate angles in standard position, coterminal angles and reference angles.

How much do you know

A. Write the letter of the correct answer.

1. What is the circumference of a circle in terms of π ?

   a. π b. 2π c. 3π d. 4π

2. An acute angle between the terminal side and the x-axis is called ______

   a. coterminal b. reference c. quadrantal d. right

3. 60 ° in radian measure is equal to

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{6}$

4. 2.5 rad express to the nearest seconds is equal to

a. 143° 14′ 24″ b. 143° 14′ 26″ c. 43° 14′ 26″ d. 43° 14′ 27″

5. What is the measure of an angle subtended by an arc that is 7 cm if the radius of the circle is 5 cm?

a. 1. 4 rad b. 1.5 rad c. 1.6 rad d. 1.7 rad

6. Point M ($\frac{24}{25}$, y) lie on the unit circle and M is in Q II. What is the value of y?

a. $\frac{6}{25}$ b. $\frac{-6}{25}$ c. $\frac{7}{25}$ d. $\frac{-7}{25}$

7. What is the measure of the reference angle of a 315^{o} angle?

a. 45^{o} b. 15^{o} c. -45^{o} d. -15^{o}

8. In which quadrant does the terminal side of $\frac{5\pi}{6}$ lie?

a. I b. II c. III d. IV

9. A unit circle is divided into 10 congruent arcs. What is the length of each arc?

a. $\frac{\pi}{10}$ b. $\frac{\pi}{5}$ c. $\frac{2\pi}{5}$ d. 10π

B. Solve:

10. The minute hand of the clock is 12 cm long. Find the length of the arc traced by the minute hand as it moved from its position at 3:00 to 3:40.

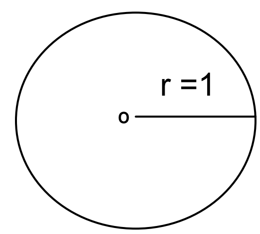
What you will do

Lesson 1

The Unit Circle

A unit circle is defined as a circle whose radius is equal to one unit and whose center is at the origin. Every point on the unit circle satisfies the equation $x^2 + y^2 + 1$.

The figure below shows a circle with radius equal to 1 unit. If the circumference of a circle is defined by the formula $c = 2\pi r$ and $r = 1$, then $c = 2\pi$ or 360° or 1 revolution.

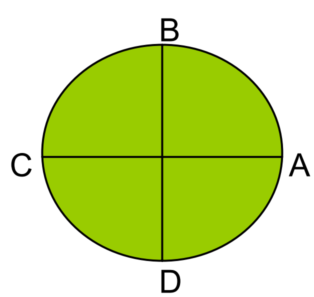


If $2\pi = 360°$, then $\pi = 180°$ or one-half revolution.

Example:

1. Imagine the Quezon Memorial Circle as a venue for morning joggers. The maintainers have placed stopping points where they could relax.

If each jogger starts at Point A, the distance he would travel at each terminal point is shown in table below.



Stopping Point	B	C	D	A
Distance or Arclength	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

This illustrates the circumference of the unit circle 2π when divided by 4: will give $\frac{2\pi}{4} = \frac{\pi}{2}$, the measure of each arc.

Similarly, the measure of each arc of a unit circle divided into:

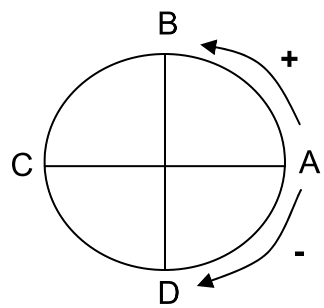
a. 6 congruent arcs = $\frac{2\pi}{6}$ = $\frac{\pi}{3}$

b. 8 congruent arcs = $\frac{2\pi}{8}$ = $\frac{\pi}{4}$

c. 12 congruent arcs = $\frac{\pi}{6}$

These measurements are called *arclengths*.

Let's go back to the unit circle which we divided into 4 congruent arcs. From A, the length of each arc in each terminal points is given as:



B: $\frac{\pi}{2}$

C: $\frac{2\pi}{2}$ = π

D: $\frac{3\pi}{2}$

A: $\frac{4\pi}{2}$ = 2π

This is true in a counterclockwise rotation. If the rotation goes clockwise, the arclengths would be negative.

Thus, the arclengths of the terminal points in a clockwise direction would yield:

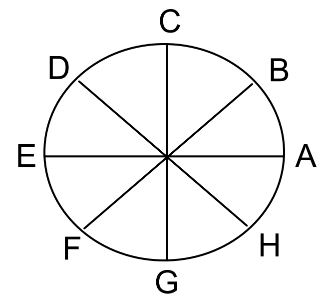
D = $-\frac{\pi}{2}$ B = $-\frac{3\pi}{2}$

C = $-\pi$ A = -2π

We call these measurements as directed arclengths.

2. Suppose a point is allowed to move around the circle starting from point A, find the arclength of each terminal point.

The unit circle is divided into 8 congruent arcs. Therefore, each arc measures $\frac{\pi}{4}$.



A counterclockwise move that terminates at:

Terminal pt.	Arclength
B	$\frac{\pi}{4}$
C	$\frac{2\pi}{4} or \frac{\pi}{2}$
D	$\frac{3\pi}{4}$
E	$\frac{4\pi}{4} or \pi$
F	$\frac{5\pi}{4}$
G	$\frac{6\pi}{4} or \frac{3\pi}{2}$
H	$\frac{7\pi}{4}$
A	$\frac{8\pi}{4} or 2\pi$

A clockwise move that terminates at:

Terminal pt.	Arclength
H	$-\frac{\pi}{4}$
G	$-\frac{2\pi}{4} or -\frac{\pi}{2}$
F	$-\frac{3\pi}{4}$
E	$-\frac{4\pi}{4} or -\pi$
D	$-\frac{5\pi}{4}$
C	$-\frac{6\pi}{4} or -\frac{3\pi}{2}$
B	$-\frac{7\pi}{4}$
A	$-\frac{8\pi}{4} or -2\pi$

A rotation can be repeated. For example a two complete rotation is equal to 4π. A one and a half revolution is equal to 3π.

An arclength of $\frac{9\pi}{4}$ will also be at terminal point B. This is also equal to $2\pi + \frac{\pi}{4}$.

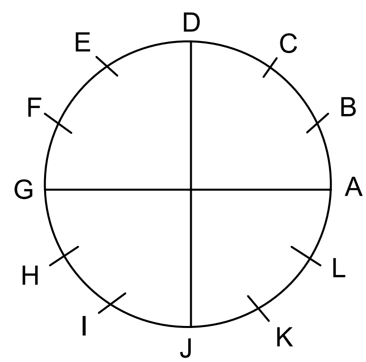
Try this out

A. Find the length of each arc of a unit circle divided into:

1. 10
2. 14
3. 16
4. 18
5. 20
6. 24

B. Given the unit circle: Identify the terminal points of each arclength:

1. $\frac{\pi}{3}$
2. $\frac{7\pi}{6}$
3. $\frac{11\pi}{6}$
4. $\frac{-7\pi}{6}$
5. $\frac{-11\pi}{6}$
6. $\frac{5\pi}{6}$
7. $\frac{-5\pi}{6}$
8. $\frac{3\pi}{2}$
9. π
10. $-\pi$
11. -2π
12. $\frac{-\pi}{3}$
13. $\frac{-2\pi}{3}$
14. $\frac{\pi}{2}$
15. $-\frac{\pi}{2}$



Lesson 2

Conversion of Degree to Radian and Vice Versa

Before discussing conversion of angle measures, you have to understand that there are two unit of angle measure that are commonly used:

1. Degree measure
2. Radian measure.

A complete revolution is divided into 360 equal parts. A degree is subdivided to minutes and seconds.

1 rev = 360°	° is the symbol for degrees
1° = 60′	′ is the symbol for minutes
1′ = 60″	″ is the symbol for seconds

For all circles, the radian measure of the circumference is 2π radians. But the angle has a measure of 360°.

hence, 2π rad = 360°

π rad = 180°

1 rad = $\frac{180}{\pi}$ or 57.296°

1° = $\frac{\pi}{180}$ rad or 0.017453 rad

Now, you are to convert degrees to radians. To convert from degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$. Then simplify.

Examples:

Convert the measure of the following angles from degrees to radians.

1. 70° = 70° x $\frac{\pi}{180}$ = $\frac{7\pi}{18}$ rad

2. -225° = -225° x $\frac{\pi}{180}$ = $\frac{-5\pi}{4}$ rad

3. 90° = 90° x $\frac{\pi}{180}$ = $\frac{\pi}{2}$ rad

4. 135° = 135° x $\frac{\pi}{180}$ = $\frac{3\pi}{4}$ rad

5. 270° = 270° x $\frac{\pi}{180}$ = $\frac{3\pi}{2}$ rad

To convert from radians to degree, multiply the number of radians by $\frac{180}{\pi}$. Then simplify.

Examples:

Express each radian measure in degrees

1. $\frac{2\pi}{4} = \frac{2\pi}{4} \times \frac{180}{\pi} = 90°$

2. $\frac{5\pi}{3} = \frac{5\pi}{3} \times \frac{180}{\pi} = 300°$

3. $\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi} = 120°$

4. $\frac{-11\pi}{6} = \frac{-11\pi}{6} \times \frac{180}{\pi} = -330°$

5. $\frac{-23\pi}{3} = \frac{-23\pi}{3} \times \frac{180}{\pi} = 1380°$

Try this out

A. Convert the following to radian measure:

1. 60°
2. 150°
3. 240°
4. 780°
5. -300°
6. -366°
7. 22.5°
8. 720°
9. 225°
10. 612°

B. Express each radian measure in degrees:

1. $\frac{7\pi}{2}$

2. $\frac{13\pi}{6}$

3. $\frac{20\pi}{3}$

4. $\frac{12\pi}{5}$

5. $\frac{7\pi}{2}$

6. $\frac{-7\pi}{5}$

7. $\frac{-5\pi}{9}$

8. $\frac{-23\pi}{3}$

9. $\frac{-4\pi}{5}$

10. $\frac{-7\pi}{4}$

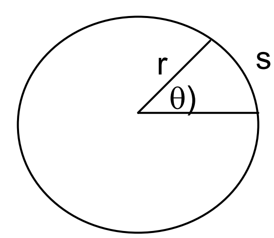
Lesson 3

Angles Intercepting an Arc

A radian is defined as the measure of an angle intercepting an arc whose length is equal to the radius of the circle. An arc length is the distance between two points along a circle expressed in linear units.

angle in radian $= \frac{arclength}{radius\ of\ the\ circle}$

or $\theta = \frac{s}{r}$



You can now use this knowledge to solve problems.

Examples:

1. A wheel of radius 80 cm rolls along the ground without slipping and rotates through an angle of 45°. How far does the wheel move?

   Solution: Use the formula $\theta = \frac{s}{r}$ to solve for the distance *s*.

   Let: radius = 80 cm $\theta = 45°$

   Convert 45° to π radians:

   $$45° \times \frac{\pi}{180} \text{ rad} = \frac{\pi}{4} \text{ rad}$$

   $$\theta = \frac{s}{r}$$

   $$\frac{\pi}{4} \text{ rad} = \frac{s}{80}$$

   $$s = \frac{\pi}{4} \text{ rad} \times 80$$

   $$s = 20\pi$$

2. The minute hand of a clock is 5 cm long. How far does the tip of the hand travel in 35 min?

   Solution:

   Arc length formula $= \frac{\deg ree}{180} (2\pi r)$

   360° in 60 min time or $\frac{360}{60} = 6°$

   35 min $\Rightarrow$ 35 x 6° = 120°

   $$L = \frac{120}{360} (2) (3.1416) (5 \text{ cm})$$

   $$= 18.33 \text{ cm}$$

Try this out

Solve the following:

1. The pendulum of a clock swings through an angle of 0.15 rad. If it swings a distance of 30 cm, what is the length of the pendulum?

2. The minute hand of the clock is 10 cm long. How far does the tip of the hand move after 12 minutes?

3. An arc 15 cm long is measured on the circumference of a circle of radius 10 cm. Find an angle subtended at the center.

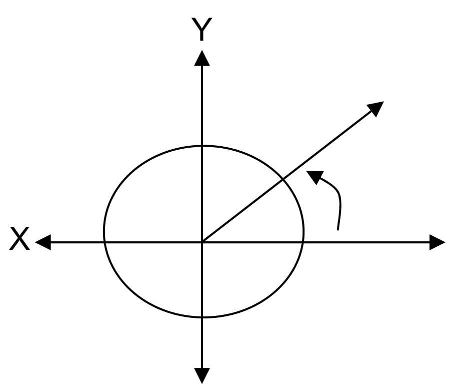
Lesson 4

Rotations Along the Unit Circle

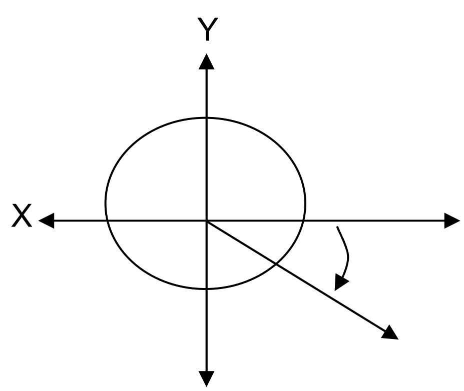
An angle can be thought of as the amount of rotation generated when a ray is rotated about its endpoints. The initial position of the ray is called the initial side of the angle and the position of the ray at the endpoint is called terminal side. A clockwise rotation generates a negative angle while a counterclockwise rotation generates a positive angle.

Imagine the terminal side of an angle whose terminal side is on the positive x-axis being rotated along the unit circle.

Positive angle



Negative angle



Example 1:

Illustrate 1. $\frac{5\pi}{2}$ radians 5. 30°

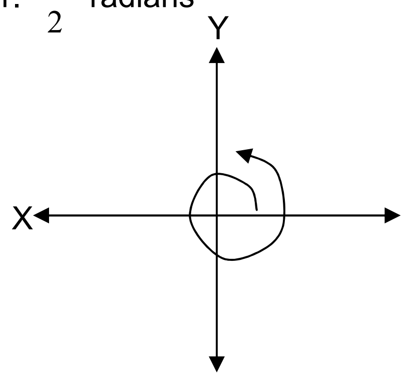
2. $\frac{9\pi}{4}$ radians 6. -90°

3. 3π radians 7. -500°

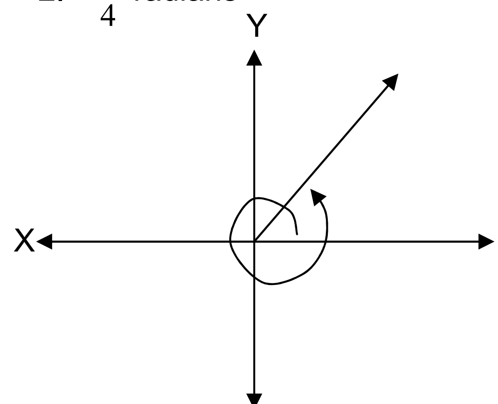
4. $\frac{-13\pi}{4}$ radians 8. 270°

The positive side of the x-axis is the initial side

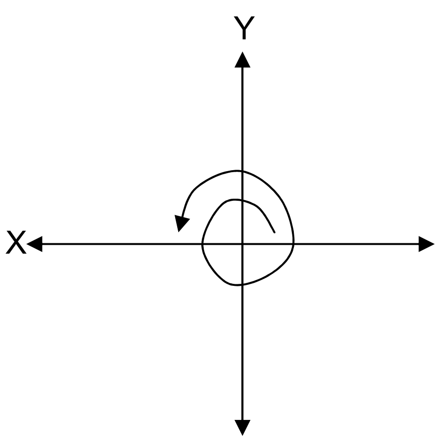
1. $\frac{5\pi}{2}$ radians



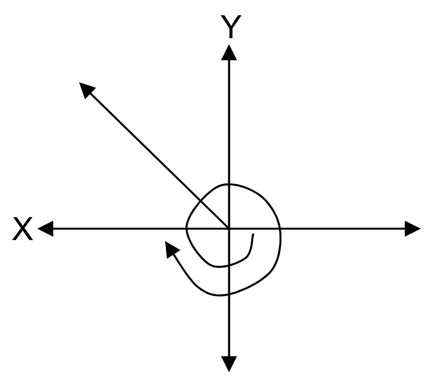
2. $\frac{9\pi}{4}$ radians



3. 3π radians



4. $\frac{-13\pi}{4}$ radians



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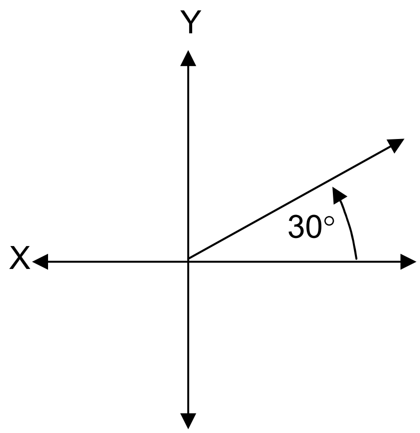

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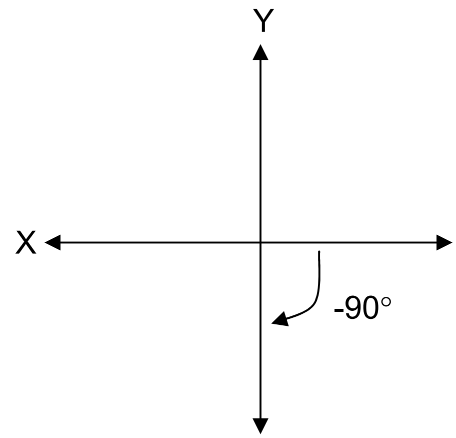
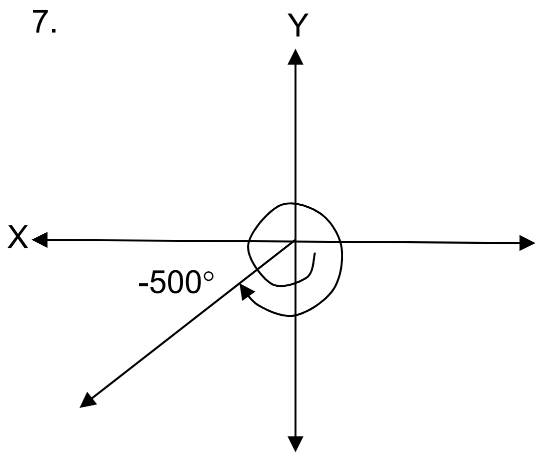
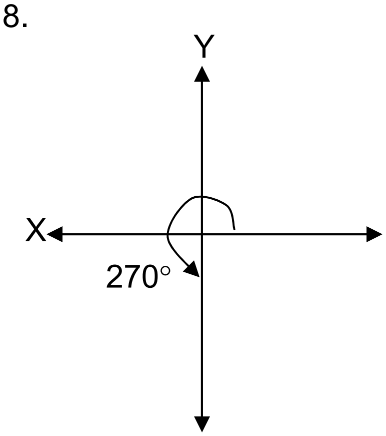


7.



8.



Example 2:

How many degrees is the angle formed when the rotating ray makes

a. 3 complete counterclockwise turns?

b. $2\frac{5}{6}$ complete clockwise turns?

Solutions:

a. 3 (360)° = 1080°

b. $2\frac{5}{6}$ (-360°) = -1020°

Try this out

A. Draw an arc whose length is:

1. 4π units
2. $\frac{5\pi}{4}$ units
3. -3π units
4. $\frac{-3\pi}{2}$ units
5. $\frac{7\pi}{12}$ units

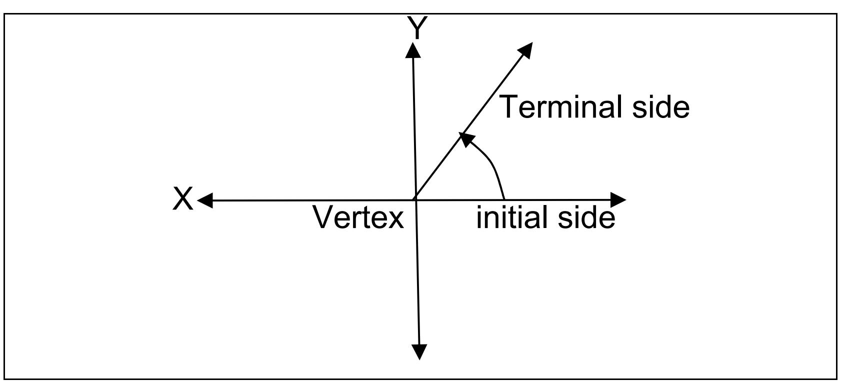
B. Draw the following angle measures.

1. 115°
2. -250°
3. -620°
4. 300°

Lesson 5

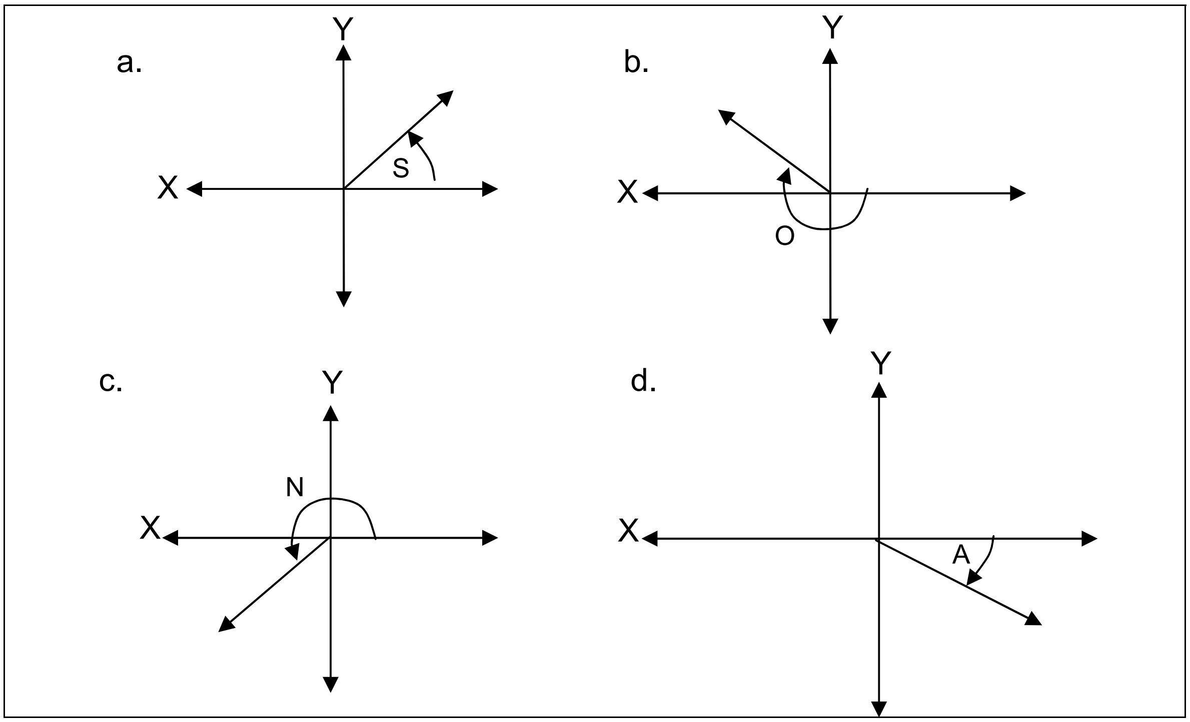
Angles

An angle whose vertex lies at the origin of the rectangular coordinate system and whose initial side is positive along the positive x-axis is said to be in **standard position**.

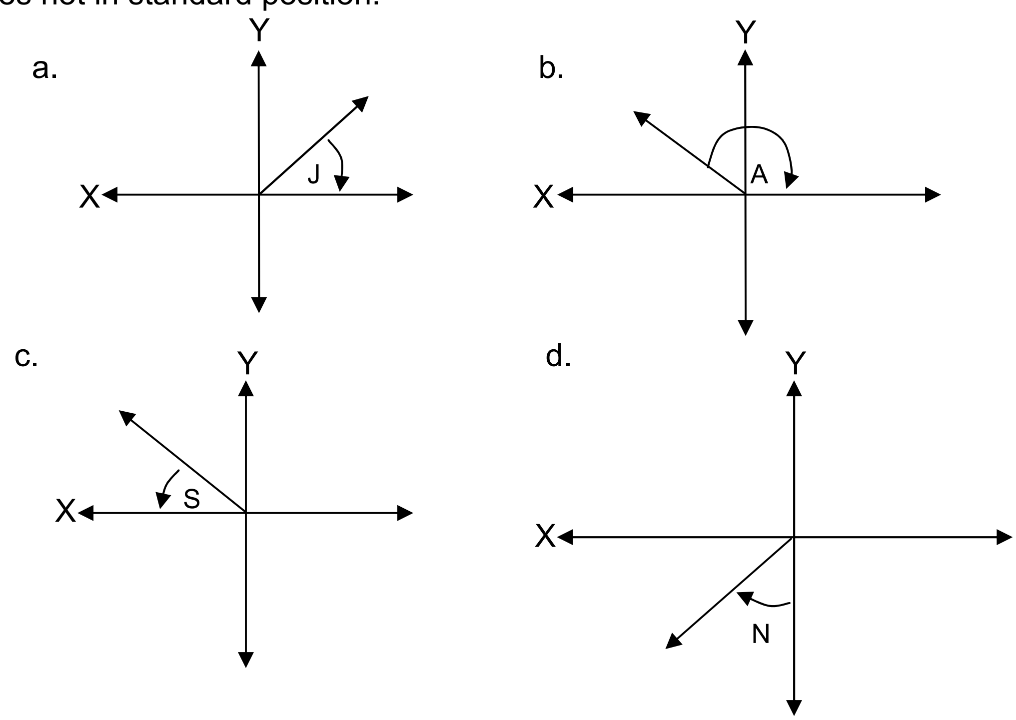


Angles in standard position.

Examples:



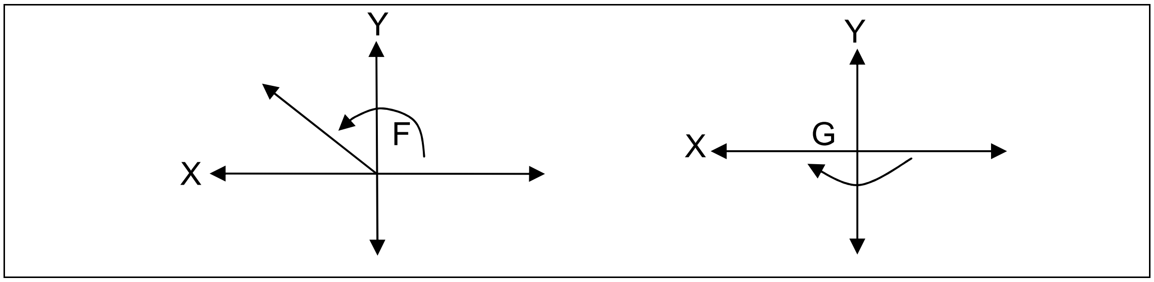
Angles not in standard position:



Quadrantal Angles:

A quadrantal angle is an angle in standard position and whose terminal side lies on the x-axis or y-axis.

Example:

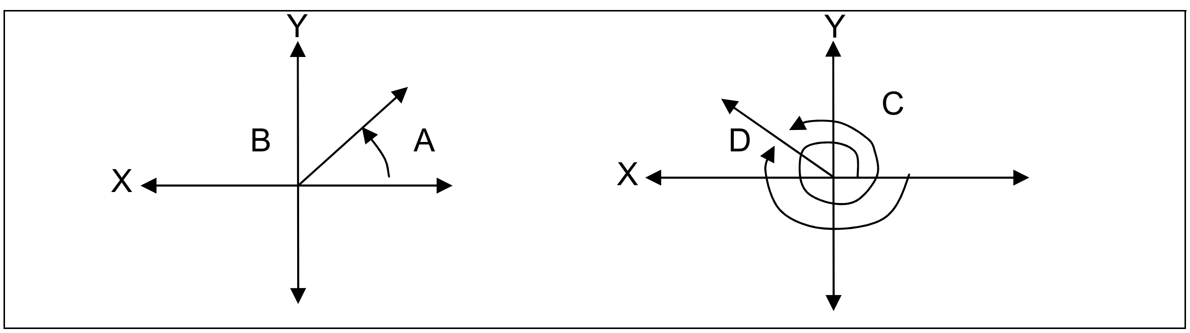


$\angle$F is not a quadrantal angle, since the terminal side does not lie on the x – axis or y - axis.

$\angle$G is a quadrantal angle since its terminal side lie on the x - axis.

Coterminal Angles:

Coterminal angles are angles having the same initial side and the same terminal side.



Examples:

Determine the measure of the smallest positive angle coterminal with:

a. 65°

b. 128°

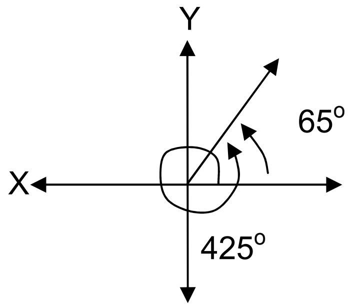
c. -213°

d. 654°

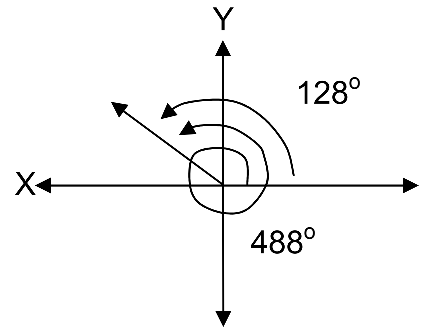
Solution:

Angles coterminal with a given angle θ may be derived using the formula $\theta + 360n$ for all integers n.

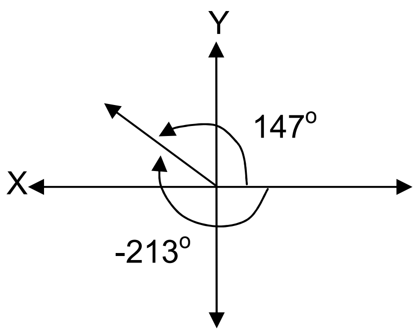
a. $65° + 360° = 425°$



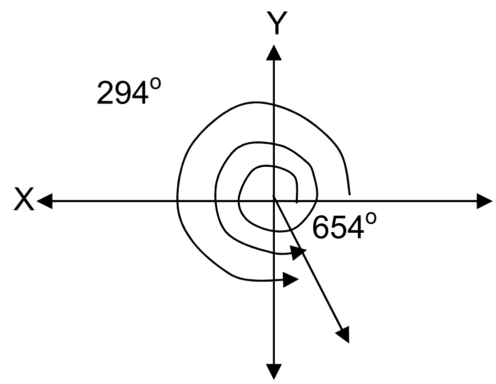
b. $128° + 360° = 488°$



c. $-213° + 360° = 147°$



d. $654° - 360° = 294°$



Finding coterminal angle less than 360°

Examples:

a. $750° = 755° - 360°(20)$
$= 35°$

b. $380° = 380° - 360°$
$= 20°$

c. $660° = 660° - 360°$
$= 300°$

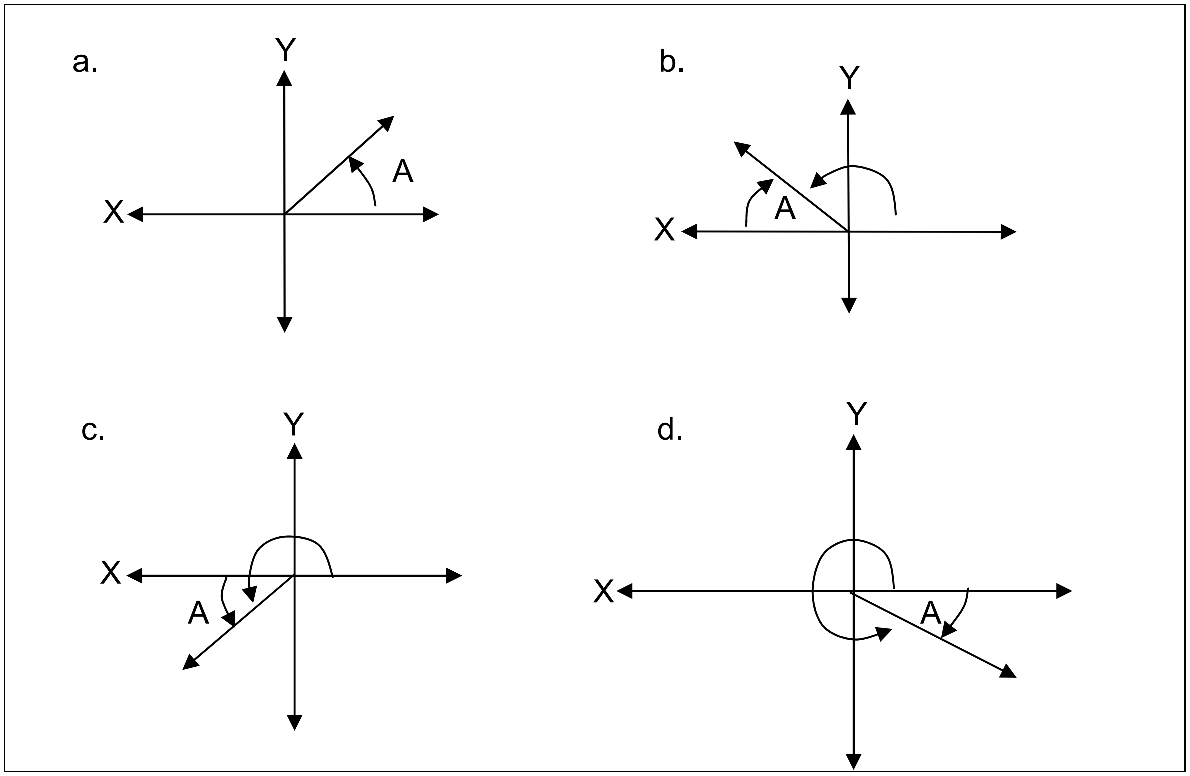
d. $820° = 820° - 360°(2)$
$= 100°$

Reference Angles

A reference angle (A) is a positive acute angle formed between the x-axis and the terminal side of a given angle.

Examples:

Let: A = reference angle



Examples:

In each of the following determine the quadrant in which the angle lies and determine the reference angle.

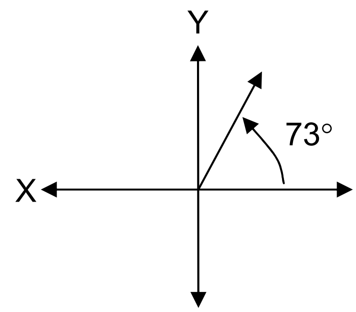
a. 73° b. 135°

c. 300° d. 920°

Solution:

The reference angle can be derived using the formula $180°n \pm \theta$.

a. 73°



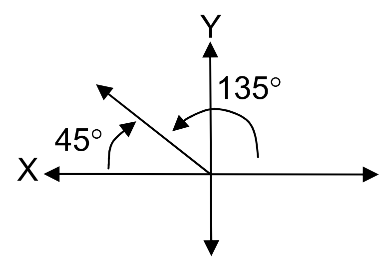
$180°n \pm \theta$

. 73° terminates in QI, hence

$180°(0) - \theta = 73°$

$\theta = 73°$, the reference angle itself

b. 135°



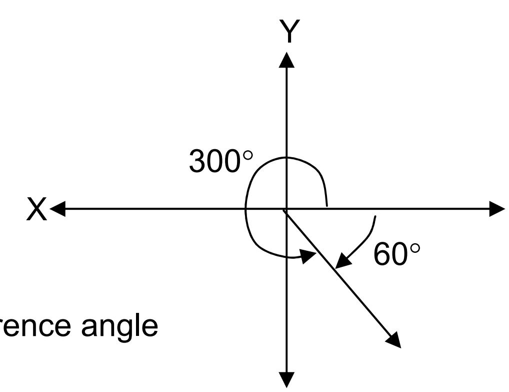
135° terminates in QII, hence

$180°(1) - \theta = 135°$

$\theta = 180° - 135°$

$\theta = 45°$ is the reference angle

c. 300°



300° terminates in Q1V , hence

$180°(2) - \theta = 300°$

$\theta = 360° - 300°$

$\theta = 60°$ is the reference angle

d. $920°$

First find the number of multiples of $180°$ in $920°$

$900°$ has 4 multiples of $180°$ and a remainder of $200°$

The terminal side of $200°$ is in QIII.

$$180°(1) - \theta = 200°$$

$$\theta = 200° - 180°$$

$$\theta = 20°$$ is the reference angle

Illustration is left for you.

Try this out

A. Determine the smallest positive coterminal angle with the given angle.

1. 57°
2. -250°
3. 94°
4. -175°
5. 116°
6. -349°
7. 100°
8. 207°
9. 185°
10. 409°

B. Determine the quadrant in which the angle lies and find the reference angle.

1. 84°
2. -140°
3. 355°
4. -365°
5. 290°
6. 480°
7. -650°
8. 740°
9. 330°
10. 204

Let's summarize

The circle of radius one with center at origin is called the **unit circle**

To convert from degrees to radians, multiply the number of degrees by $\frac{\pi}{180}$. Then simplify.

To convert from radians to degree, multiply the number of radians by $\frac{180}{\pi}$. Then simplify.

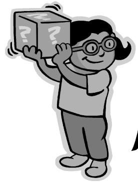
An angle is the amount of rotation where one side is called the initial side and the other is the terminal side.

An angle is in standard position if it is constructed in a rectangular coordinate system with vertex at the origin and the initial side on the positive side of the x-axis.

Coterminal angle are angles having the same initial side and the same terminal side.

Reference angle is an acute angle between the terminal side and the x-axis.

To find the reference angle, write the angle in the form $180n \pm \theta$ where θ is the reference angle.



What have you learned

Answer the following correctly:

1. A circle is divided into 6 congruent arcs. What is the measure of each arc?

2. Express 120° in radian measure.

3. What is the reference angle of -380°?

4. The coterminal angle less than 360° of 820° is __________.

5. Convert $\frac{-7\pi}{6}$ rad to degree measure.

6. On a circle of radius 20cm, the arc intercepts a central angle of $\frac{1}{5}$ rad. What is the arclength?

7. At what quadrant is the terminal side of - 1080° located?

8. How many degrees is the angle formed by a ray that makes $3\frac{1}{5}$ complete rotations counterclockwise?

9. How many degrees is the angle formed by a ray that makes $2\frac{2}{3}$ complete rotations clockwise?

10. A minute hand of a clock is 5 cm long. How far does the tip of the hand travel in 50 min?

Answer Key

How much do you know

1.	b	6.	c
2.	b	7.	a
3.	d	8.	QII
4.	a	9.	$\frac{5\pi}{20}$
5.	a	10.	20.57 cm

Try this out

Lesson 1

A.
1. $\frac{\pi}{5}$
2. $\frac{\pi}{7}$
3. $\frac{\pi}{8}$
4. $\frac{\pi}{9}$
5. $\frac{\pi}{10}$
6. $\frac{\pi}{12}$

B.
1. C
2. H
3. L
4. F
5. B
6. F
7. H
8. J
9. G
10. G
11. A
12. K
13. I
14. D
15. J

Lesson 2

A.

1. $\frac{\pi}{3}$ rad
2. $\frac{5\pi}{3}$ rad
3. $\frac{4\pi}{3}$ rad
4. $\frac{13\pi}{3}$ rad
5. $\frac{-5\pi}{3}$ rad
6. $\frac{61\pi}{30}$ rad
7. $\frac{\pi}{8}$ rad
8. 4π
9. $\frac{5\pi}{4}$
10. $\frac{17\pi}{5}$

B.

1. 63°
2. 390°
3. 1200°
4. 432°
5. 105°
6. -252°
7. -100°
8. -1300°
9. -144°
10. -315°

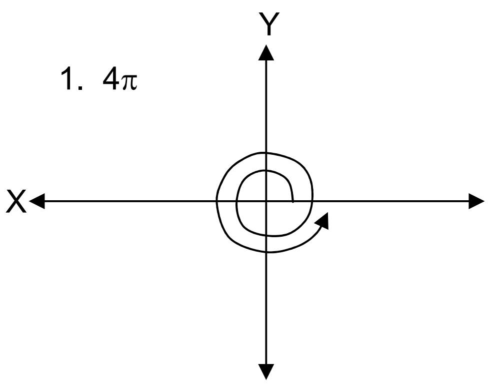
Lesson 3

1. 200 cm
2. 12.57 cm
3. 1.5 rad

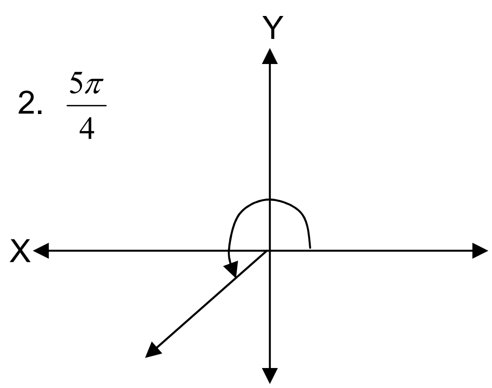
Lesson 4

A.

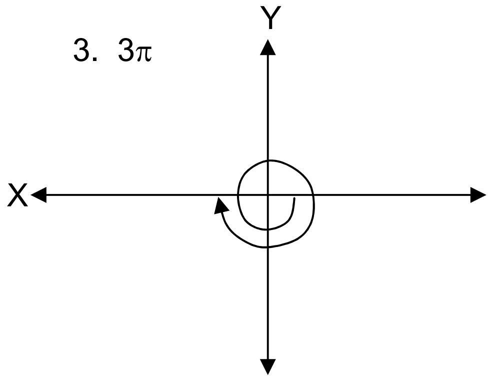
1. 4π



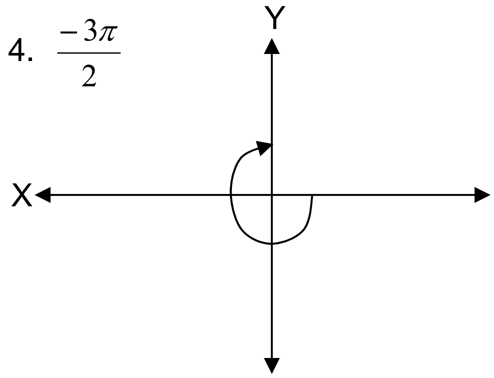
2. $\frac{5\pi}{4}$



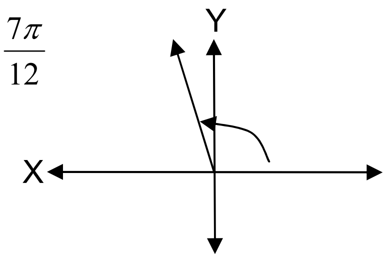
3. 3π



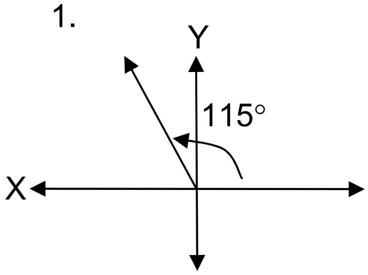
4. $\frac{-3\pi}{2}$



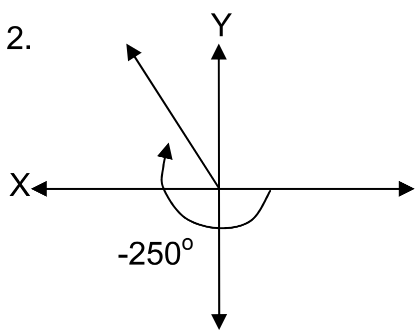
5. $\frac{7\pi}{12}$



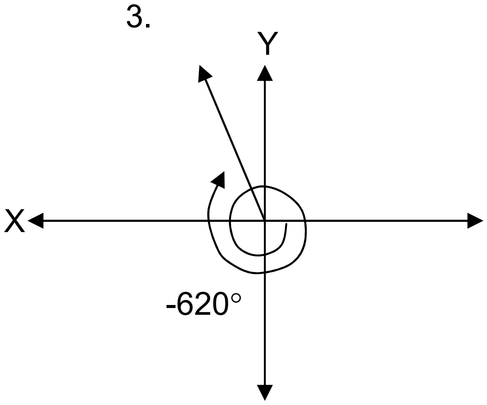
B. 1.



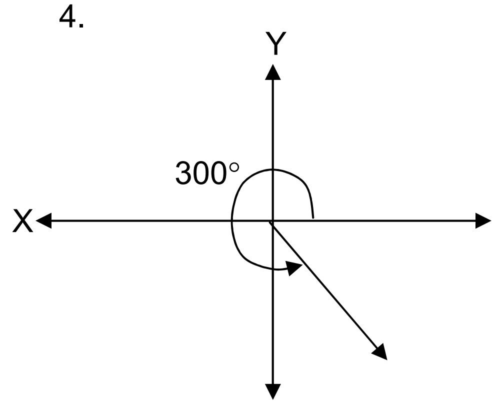
2.



3.



4.



Lesson 5

A.

1. 417°	6. 11°
2. 110°	7. 460°
3. 454°	8. 567°
4. 185°	9. 545°
5. 476°	10. 769°

B.

1. QI , 84°	6. Q1, 60°
2. QIII, 35°	7. QII, 10°
3. Q IV, 5°	8. Q II, 20°
4. QIV, 5°	9. Q1, 30°
5. Q III, 70°	10. Q III, 24°

What have you learned

1. $\frac{\pi}{3}$	6. s = 4 cm
2. $\frac{2\pi}{3}$ rad	7. Q1
3. 20°	8. -960°
4. 100°	9. 1152°
5. -21π rad	10. 26.18 cm