Module 2
Exponential Functions

What this module is about

This module is about the roots of exponential equations and zeros of exponential functions. As you go over this material, you will develop the skills in finding the roots of exponential equations and the zeros of the exponential functions using the property of equality for exponential equation and the laws of exponents.

What you are expected to learn

The module is designed for you to use the laws on exponents to find the roots of exponential equations and the zeros of exponential functions

How much do you know

1. If $3^x = 3^4$, what is x?
2. Find x if $2^{x-1} = 4$.
3. Simplify the expression $(4x^5)^2$.
4. Express $\left(\frac{4x^{-4}}{-2x^{-9}}\right)^{-2}$ without negative exponent.
5. Solve for x in the equation $2^2(5^{x+1}) = 500$
6. What are the values of x in $3^{x^2} = 9^{2x-1}$?
7. Determine the zeros of the exponential function $F(x) = 2^x$.
8. Find the zeros of $h(x) = 2^{x-3}$.
9. Where will the graph of $y = 1 + \left(\frac{1}{2}\right)^{2x+1}$ cross the x-axis?
10. What value of x will make the function value of $y = 3^{2x} - 1$ equal to 0?

What you will do

Lesson 1

The Property of Equality for Exponential Equation

An exponential equation in one variable is an equation where the exponent is the variable.

In solving exponential equations, the Property of Equality for Exponential Equation is used. It is stated as

"If a, b, and c are real numbers and $a \neq 0$, then, $a^b = a^c$ if and only if $b = c$."

Examples: Solve for the value of the variable that would make the equation true.

1. $3^x = 3^5$ — Since the bases are equal,
   $x = 5$ — the exponents must be equal too.

   Since the $x = 5$, then $3^5 = 3^5$

2. $4^{2y} = 4^8$ — Since the bases are equal,
   $2y = 8$ — the exponents must be equal too.
   $y = 4$

   Since $y = 4$, then $4^{2y} = 4^8 \longrightarrow 4^{2(4)} = 4^8 \longrightarrow 4^8 = 4^8$

3. $7^2 = 7^{z-1}$ — The bases are equal,
   $2 = z - 1$ — the exponents must be equal too.
   $z = 3$

   Since $z = 3$, then $7^2 = 7^{z-1} \longrightarrow 7^2 = 7^{3-1} \longrightarrow 7^2 = 7^2$

Try this out

Find the value of the variable that would make each equation true.

Set A

1. $3^x = 3^9$
2. $2^{3y} = 2^{12}$
3. $5^{2z} = 5^3$

4. $8^{a-2} = 8^7$
5. $12^{m+3} = 12^0$
6. $4^{4b+1} = 4^5$
7. $7^{5x+2} = 7^{4x-1}$
8. $9^{2(x-1)} = 9^{3x+1}$
9. $6^{4(2x-1)} = 6^{3(x-1)}$
10. $\pi^{x^2+2x} = \pi^{-1}$

Set B

1. $6^x = 6^{11}$
2. $9^{5y} = 9^{15}$
3. $12^{6z} = 12^3$
4. $10^{a-4} = 10^{-9}$
5. $7^{2b+2} = 7^{b-3}$
6. $2^{3n-7} = 2^{n+5}$
7. $3^{4(x+2)} = 3^{5x-2}$
8. $8^{-(3-2x)} = 8^{3(-4+x)}$
9. $4^{x^2-x} = 4^{12}$
10. $5^{4x^2-4x} = 5$

Set C

1. $4^{2x} = 4^9$
2. $3^{8y} = 3^{12}$
3. $6^{11z+2} = 6^{3z-10}$
4. $2^{4y+(2-y)} = 2^{3-5y}$
5. $8^{\sqrt{x}} = 8^5$
6. $5^{\sqrt{x+3}} = 5^2$
7. $7^{x^2} = 7^{2x-1}$
8. $9^{x^2} = 9^{x+6}$
9. $\pi^{2x^2-x} = \pi^3$

10. $\left(\sqrt{2}\right)^{2x^2+5x} = \left(\sqrt{2}\right)^3$

Lesson 2

Review of the Laws of Exponents

Let us review the Laws of Exponents for easy reference.

For any real numbers a and b, and any positive real numbers m and n,

a. $a^m a^n = a^{m+n}$

b. $(a^m)^n = a^{mn}$

c. $(ab)^n = a^n b^n$

d. $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$

f. $a^0 = 1$

Examples:

Simplify each expression. Express answers with positive exponents:

1. $(x^3)(x^6)$

   $(x^3)(x^6) = x^{3+6}$

   $= x^9$

2. $(x^{-3})^2$

   $(x^{-3})^2 = x^{(-3)(2)}$

   $= x^{-6}$

   $= \frac{1}{x^6}$

3. $(2x^{-5})^{-3}$

   $(2x^{-5})^{-3} = 2^{-3}x^{15}$

$$= \frac{x^{15}}{2^3}$$

$$= \frac{x^{15}}{8}$$

4. $\frac{16x^5}{12x^7}$

$$\frac{16x^5}{12x^7} = \frac{16}{12} \cdot \frac{x^5}{x^7}$$

$$= \frac{4}{3} \cdot x^{5-7}$$ Simplify

$$= \frac{4}{3} \cdot x^{-2}$$

$$= \frac{4}{3} \cdot \frac{1}{x^2}$$

$$= \frac{4}{3x^2}$$

5. $\left[\frac{64x^{\frac{1}{3}}}{125x^{-\frac{2}{3}}}\right]^{\frac{2}{3}}$

$$\left[\frac{64x^{\frac{1}{3}}}{125x^{-\frac{2}{3}}}\right]^{\frac{2}{3}} = \left[\frac{64}{125}\right]^{\frac{2}{3}} \cdot \left[\frac{x^{\frac{1}{3}}}{x^{-\frac{2}{3}}}\right]^{\frac{2}{3}}$$

$$= \left[\sqrt[3]{\frac{64}{125}}\right]^2 \cdot \left[x^{\frac{1}{3}+\frac{2}{3}}\right]^{\frac{2}{3}}$$ Express rational exponents in radical form and simplify.

$$= \left[\frac{4}{5}\right]^2 \cdot \left[x^{\frac{3}{3}}\right]^{\frac{2}{3}}$$

$$= \left[\frac{16}{25}\right] \cdot \left[x^1\right]^{\frac{2}{3}}$$

$$= \frac{16x^{\frac{2}{3}}}{25}$$

Try this out

Simplify each expression. Express answers with positive exponents:

Set A

1. $(x^6)(x^7)$
2. $(3x^4)(-6x^7)$
3. $(x^5)^4$
4. $(3x^3)^2$
5. $\frac{8x^9}{2x^5}$
6. $\frac{-15x^{-2}}{25x^5}$
7. $\frac{21x^6}{-14x^{-4}}$
8. $\frac{\left(-2x^5\right)^2}{\left(-4x^2\right)^3}$
9. $\left(\frac{8x^9}{2x^5}\right)^{-2}$
10. $\left[\frac{\left(-2x^2\right)^3}{\left(3x^{-3}\right)^{-2}}\right]^{-1}$

Set B

1. $(5x^{-4})(-x^2)$
2. $(-2x^{-3})(-4x^{-2})$
3. $(3x^{-4})^3$
4. $(-5x^{-4})^2$
5. $\frac{12x^8}{20x^3}$

6. $\dfrac{-33x^{-6}}{24x^{2}}$

7. $\dfrac{48x^{9}}{-80x^{-13}}$

8. $\dfrac{\left(-2x^{-4}\right)^{2}}{\left(3x^{2}\right)^{-3}}$

9. $\left(-\dfrac{6x^{5}}{8x^{9}}\right)^{-2}$

10. $\left[\dfrac{\left(4x^{2}\right)^{-1}}{\left(-2x^{-2}\right)^{-3}}\right]^{-2}$

Set C

1. $\dfrac{-27x^{6}}{21x^{4}}$

2. $\dfrac{(2x^{2})^{3}}{8x^{9}}$

3. $\dfrac{(3x^{-2})^{4}}{(9x^{3})^{-2}}$

4. $\dfrac{(8x^{6})^{2}}{(2x^{3})^{4}}$

5. $\left(\dfrac{8x^{\frac{1}{2}}}{27x}\right)^{\frac{2}{3}}$

6. $\left(\dfrac{64x^{-6}}{36x^{2}}\right)^{\frac{1}{2}}$

7. $-\left(\dfrac{32x^{9}}{-72x^{-13}}\right)^{-2}$

8. $\left[\dfrac{\left(2x^{3}\right)^{-3}}{\left(-3x^{2}\right)^{2}}\right]^{2}$

9. $\left[\left(-\frac{2x}{8x^4}\right)^{-2}\right]^{-2}$

10. $\left[\frac{\left(-2x^2\right)^{-2}}{-\left(2x^{-2}\right)^{-3}}\right]^{-3}$

Lesson 3

Finding the Roots of Exponential Equation

The Property of Equality for Exponential Equation which is also known as Equating-Exponents Property implies that in an exponential equation, if the bases are equal, the exponents must also be equal.

Examples:

Solve each exponential equation.

1. $2^{4x+2} = 8^{x-2}$

Use laws of exponents to make the bases equal. Then apply the Equating Exponents Property.

$$2^{4x+2} = 8^{x-2}$$
$$2^{4x+2} = 2^{3(x-2)}$$
$$4x + 2 = 3(x - 2)$$
$$4x + 2 = 3x - 6$$
$$x = -8$$

2. $9^{-x} = \frac{1}{27}$

Use laws of exponents to make the bases equal. Then apply the Equating Exponents Property.

$$9^{-x} = \frac{1}{27}$$
$$3^{2(-x)} = 27^{-1}$$

$3^{-2x} = (3^3)^{-1}$

$3^{-2x} = 3^{-3}$

$-2x = -3$

$x = \frac{-3}{-2}$

$x = \frac{3}{2}$

3. $2^{x^2-5x} = \frac{1}{16}$

Use laws of exponents to make the bases equal. Then apply the Equating Exponents Property.

$2^{x^2-5x} = \frac{1}{16}$

$2^{x^2-5x} = 16^{-1}$

$2^{x^2-5x} = \left(2^4\right)^{-1}$

$2^{x^2-5x} = 2^{-4}$

$x^2 - 5x = -4$

$x^2 - 5x + 4 = 0$

$(x - 1)(x - 4) = 0$

$x = 1$ or $x = 4$

Try this out

Solve for x in each exponential equation.

Set A

1. $2^x = 128$
2. $5^x = 125$
3. $2^x = \frac{1}{2}$
4. $9^{2x} = 27$
5. $2^{x-1} = 32$
6. $1000^x = 100^{2x-5}$

7. $4^{3x + 1} = 8^{x - 1}$
8. $9^{3x} = 27^{x - 2}$
9. $8^{x - 1} = 16^{3x}$
10. $9^{3x + 1} = 27^{3x + 1}$

Set B

1. $9^x = 27$
2. $4^x = 128$
3. $4^x = \frac{1}{2}$
4. $3^{x + 2} = 27$
5. $25^{x + 1} = 125^x$
6. $4^x = 0.0625$
7. $3^{-x} = \frac{1}{243}$
8. $8^{x + 3} = \frac{1}{16}$
9. $\left(\frac{1}{3}\right)^{-x} = 27^{2x+1}$
10. $\left(\frac{1}{4}\right)^{3-2x} = 8^{-x}$

Set C

1. $\left(\frac{1}{2}\right)^{x} = 4$
2. $4^x = \frac{1}{8}$
3. $5^{x + 3} = 25$
4. $9^x = 27^{x + 1}$
5. $7^{3x + 2} = 49^x$
6. $121^{3x} = 11^{x - 1}$
7. $3^{\frac{x+3}{3}} = 81^{5+x}$
8. $\left(\frac{16}{25}\right)^{x^2+x} = 1$

9. $\left(5^{x}\right)^{-\frac{2}{3}} = 0.04$

10. $\left(\frac{1}{8}\right)^{\frac{1}{3}x} = \frac{1}{2}$

Lesson 4

Determining the Zeros of Exponential Functions

As in other function, the zero of an exponential function refers to the value of the independent variable x that makes the function 0. Graphically, it is the abscissa of the point of intersection of the graph of the exponential function and the x-axis.

To find the zero of an exponential function f(x), equate f(x) to 0 and solve for x.

Examples:

Determine the zero of the given exponential function.

1. $f(x) = 2^x$

   To find the zero of the function, equate it to 0 and solve for x.

   $f(x) = 2^x = 0$
   $2^x = 0$

   The resulting equation suggests that f(x) has no zero since no real value of x will make $2^x = 0$ a true statement.

2. $g(x) = 3^{4x-8} - 1$

   To find the zero of the function, equate it to 0 and solve for x.

   $g(x) = 3^{4x-8} - 1 = 0$
   $3^{4x-8} - 1 = 0$
   $3^{4x-8} = 1$
   $3^{4x-8} = 3^0$
   $4x - 8 = 0$
   $4x = 8$

$$x = 2$$

The zero of g(x) is 2.

3. $h(x) = \left(\frac{1}{3}\right)^{2x+3} + 9$

To find the zero of the function, equate it to 0 and solve for x.

$$h(x) = \left(\frac{1}{3}\right)^{2x+3} - 9 = 0$$

$$\left(\frac{1}{3}\right)^{2x+3} - 9 = 0$$

$$\left(\frac{1}{3}\right)^{2x+3} = 9$$

$$\left(\frac{1}{3}\right)^{2x+3} = 3^2$$

$$\left(3^{-1}\right)^{2x+3} = 3^2$$

$$(3)^{-2x-3} = 3^2$$

$$-2x - 3 = 2$$

$$-2x = 5$$

$$x = -\frac{5}{2}$$

The zero of h(x) is $-\frac{5}{2}$.

4. $y = 2^{2x + 3} - \left(\frac{1}{512}\right)^{x-1}$

To find the zero of the function, equate it to 0 and solve for x.

$$y = 2^{2x + 3} - \left(\frac{1}{512}\right)^{x-1} = 0$$

$$2^{2x + 3} - \left(\frac{1}{512}\right)^{x-1} = 0$$

$$2^{2x+3} = \left(\frac{1}{512}\right)^{x-1}$$

$$2^{2x+3} = \left(512^{-1}\right)^{x-1}$$

$$2^{2x+3} = \left[\left(2^{9}\right)^{-1}\right]^{x-1}$$

$$2^{2x+3} = \left(2^{-9}\right)^{x-1}$$

$$2^{2x+3} = \left(2\right)^{-9x+9}$$

$$2x + 3 = -9x + 9$$

$$11x = 6$$

$$x = \frac{6}{11}$$

The zero of y is $\frac{6}{11}$.

Try this out

Determine the zeros of the given polynomial function.

Set A

1. $f(x) = 3^{x}$
2. $g(x) = 3^{x+2} - 27$
3. $h(x) = 4^{x-1} - 64$
4. $p(x) = (7^{-x})^{2} - 343$
5. $F(x) = (0.2)^{x-2} - 5^{4x}$
6. $G(x) = 8^{2x} - 2^{2x+1}$
7. $H(x) = \left(\frac{1}{2}\right)^{x-3} - 8^{1-x}$
8. $P(x) = 4^{x(x-5)} - \left(\frac{1}{16}\right)^{3}$
9. $y = \left(\sqrt{27}\right)^{x-3} - 3^{3x-1}$

10. $y = \left(\frac{2}{3}\right)^{4x+3} - \frac{16}{81}$

Set B

1. $f(x) = 2^{2x-1}$
2. $g(x) = 16^{5+x} - 4$
3. $h(x) = 10^{3x} - (0.001)^{x+3}$
4. $p(x) = 2^{x(x-3)} - \frac{1}{16}$
5. $F(x) = \left(\frac{1}{64}\right)^{3x-4} - 2^{3-2x}$
6. $G(x) = \sqrt{343} - 49^{2x-1}$
7. $H(x) = \left(\sqrt{125}\right)^{3x-1} - \left(\frac{1}{5}\right)^{x-2}$
8. $P(x) = \left(\frac{2}{3}\right)^{x-3} - \frac{32}{243}$
9. $y = \left(\frac{3}{5}\right)^{2x+1} - \frac{27}{125}$
10. $y = \left(\frac{4}{7}\right)^{3x-1} - \left(\frac{49}{16}\right)^{x}$

Set C

1. $f(x) = 9^{2x+2} - 27$
2. $g(x) = 125^{3x+7} - \left(\frac{1}{5}\right)^{x-3}$
3. $h(x) = 2^{5x+1} - 8^{2x-3}$
4. $p(x) = 27^{5x-6} - 9^{7x+3}$
5. $F(x) = 81^{\frac{x}{4}} - \frac{1}{27}$
6. $G(x) = (9^x)(3^x) - 243$
7. $H(x) = \left(\frac{3}{2}\right)^{x+5} - \frac{729}{64}$

8. P(x) = $\left(\frac{4}{3}\right)^{2x-3} - \frac{27}{64}$

9. y = $\left(\frac{2}{5}\right)^{3x+1} - \frac{625}{16}$

10. y = $\left(\frac{3}{2}\right)^{x-3} - \frac{4}{9}$

Let's summarize

1. An exponential equation is an equation where the variable is the exponent.

2. In solving exponential equations, the Property of Equality for Exponential Equation is used. Stated as follows: "If a, b, and c are real numbers and $a \neq 0$, then, $a^b = a^c$ if and only if b = c."

3. For any real numbers a and b, and any positive real numbers m and n,

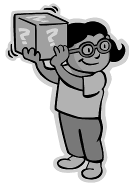
   a. $a^m a^n = a^{m+n}$
   b. $(a^m)^n = a^{mn}$
   c. $(ab)^n = a^n b^n$
   d. $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
   e. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
   f. $a^0 = 1$

4. The Property of Equality for Exponential Equation which is also known as Equating-Exponents Property implies that in an exponential equation, if the bases are equal, the exponents must also be equal.

5. The zero of an exponential function refers to the value of the independent variable x that makes the function 0.

6. Graphically, the zero of an exponential function is the abscissa of the point of intersection of the graph of the exponential function and the x-axis.

7. To find the zero of an exponential function f(x), equate f(x) to 0 and solve for x.



What have you learned

1. If $2^{2x+1} = 2^5$, what is x?
2. Find x if $4^{x+1} = 2$.
3. Simplify the expression $(5x^4)^3$.
4. Express $\left(\frac{9x^{-3}}{-36x^7}\right)^{-2}$ without negative exponent.
5. Solve for x in the equation $9(4^{2x+1}) = 36^x$
6. What are the values of x in $4^{x^2} = 8^{2x-1}$?
7. Determine the zeros of the exponential function $F(x) = 5^{x-2}$.
8. Find all the zeros of $f(x) = 7^{x-3} - 49$.
9. Where will the graph of $g(x) = \left(\frac{1}{4}\right)^x - \left(\frac{1}{2}\right)^{2x+1}$ cross the x-axis?
10. What value of x will make $h(x) = 32^{2x} - \left(\frac{1}{16}\right)^{-2x}$ equal to 0?

Answer Key

How much do you know

1. 4
2. 3
3. $16x^{10}$
4. $\frac{4}{x^{10}}$
5. 2
6. $2 \pm \sqrt{2}$
7. no zero
8. 5
9. 0 and $\frac{3}{2}$
10. 0

Try this out

Lesson 1

Set A

1. 9
2. 4
3. $\frac{3}{5}$
4. 9
5. -3
6. 1
7. -3
8. -3
9. $\frac{1}{5}$
10. -1

Set B

1. 11
2. 3
3. $\frac{1}{2}$
4. -5
5. -5
6. 6
7. 10
8. 9
9. 4 and -3
10. $\frac{2 \pm \sqrt{2}}{2}$

Set C

1. $\frac{9}{2}$
2. $\frac{3}{2}$
3. $-\frac{3}{2}$
4. $\frac{1}{8}$
5. 25
6. 1
7. 1
8. 3 and -2
9. $\frac{3}{2}$ and -1
10. $\frac{1}{2}$ and -3

Lesson 2

Set A

1. x^{13}
2. $-18x^{11}$
3. x^{20}
4. $9x^{6}$
5. $4x^{4}$
6. $-\frac{3}{5x^7}$
7. $-\frac{3x^{10}}{2}$
8. $-\frac{x^4}{4}$
9. $\frac{1}{16x^8}$
10. $-\frac{1}{72}$

Set B

1. $-\frac{5}{x^2}$
2. $\frac{8}{x}$
3. $\frac{27}{x^{12}}$
4. $\frac{25}{x^8}$
5. $\frac{3x^5}{5}$
6. $-\frac{11}{8x^8}$
7. $-\frac{3x^{22}}{5}$

8. $\frac{108}{x^2}$

9. $\frac{16x^8}{9}$

10. $\frac{x^{16}}{4}$

Set C

1. $-\frac{9x^2}{7}$

2. $\frac{1}{x^3}$

3. $\frac{6561}{x^2}$

4. 4

5. $\frac{4}{9}x^{\frac{1}{3}}$

6. $\frac{4}{3x^4}$

7. $-\frac{81}{16x^{31}}$

8. $\frac{1}{5184x^{26}}$

9. $\frac{1}{4x^{12}}$

10. $-\frac{x^{20}}{8}$

Lesson 3

Set A

1. 7
2. 3
3. -1
4. $\frac{3}{4}$
5. 6
6. 10
7. $-\frac{1}{5}$
8. -2
9. $-\frac{1}{3}$
10. $-\frac{1}{3}$

Set B

1. $\frac{3}{2}$
2. $\frac{7}{2}$
3. $-\frac{1}{2}$
4. 1
5. 2
6. -2
7. 5
8. $-\frac{13}{3}$
9. $-\frac{3}{5}$

10. $\frac{6}{7}$

Set C

1. -2
2. $-\frac{3}{2}$
3. -1
4. -3
5. -2
6. $-\frac{1}{5}$
7. $-\frac{57}{11}$
8. 0 and -1
9. 3
10. -1

Lesson 4

Set A

1. No rational zero
2. 1
3. 4
4. $-\frac{3}{2}$
5. $\frac{2}{5}$
6. $\frac{1}{4}$
7. 0
8. 2 and 3
9. $-\frac{7}{3}$

10. $\frac{1}{4}$

Set B

1. No rational zero
2. $-\frac{9}{2}$
3. $-\frac{3}{2}$
4. $\frac{3 \pm i\sqrt{7}}{2}$
5. $-\frac{27}{16}$
6. $\frac{7}{8}$
7. $\frac{7}{11}$
8. 8
9. 1
10. $\frac{1}{5}$

Set C

1. $-\frac{1}{4}$
2. $-\frac{9}{5}$
3. 10
4. 24
5. -3
6. $\frac{5}{3}$
7. 1
8. 0

9. $-\frac{5}{3}$
10. 1

How much have you learned

1. 2
2. $-\frac{1}{2}$
3. $125x^{12}$
4. $16x^{20}$
5. -1
6. $\frac{3 \pm \sqrt{3}}{2}$
7. none
8. 5
9. It will not cross the x-axis
10. 0