Module 3
Polynomial Functions

What this module is about

This module is about graphs of polynomial functions of degree greater than two. The graph of a first degree-polynomial is a line. The graph of a second-degree polynomial is a parabola. The graph of a third degree- polynomial typically has both a minimum point and a maximum point. The number of maximum and minimum points is at most one less than the degree of the polynomial. The graph of a polynomial function of degree n has n – 1 turning points.

The lessons were presented in a very simple way so it will be easy for you to understand and be able to do the graphs of the polynomial functions of degree greater than two without difficulty.

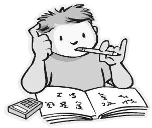
What you are expected to learn

This module is designed for you to:

1. identify the function represented by a graph.

2. draw the graphs of polynomial functions of degree greater than 2.

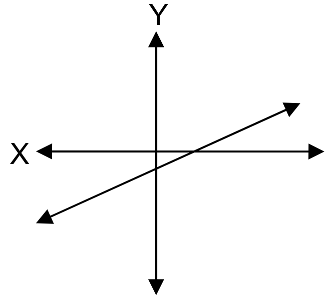
3. identify the degree of a polynomial function, its zeros, x-intercepts and turning points based on its equation and graph.



How much do you know

A. Identify whether the function represented by the graph below is linear, quadratic or polynomial.

1.



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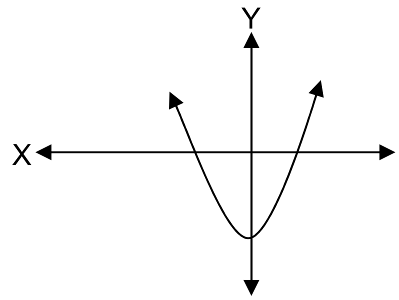

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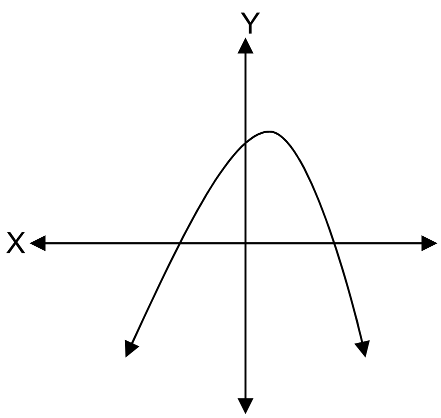

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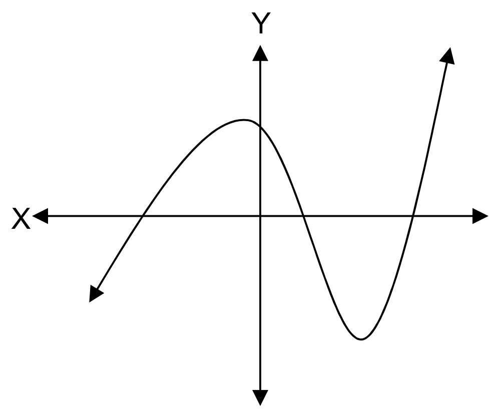

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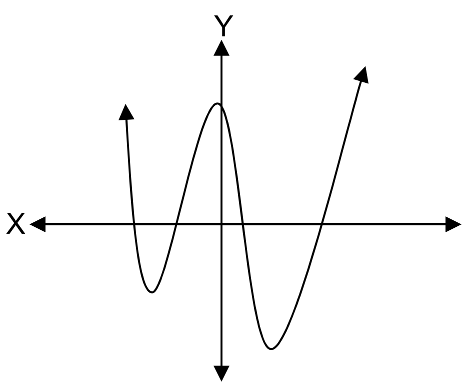
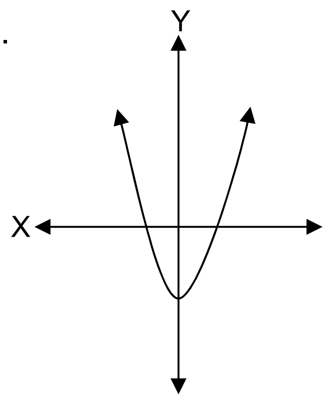


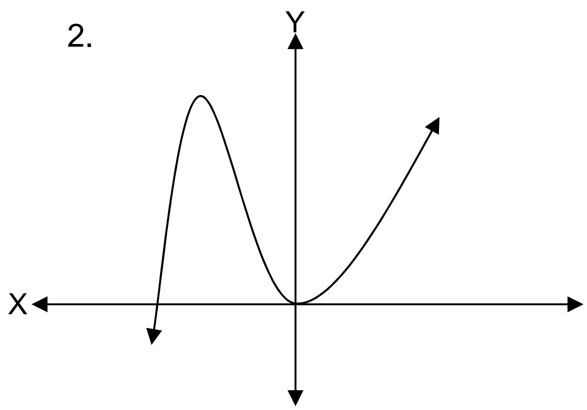
B. From the graph and equation, find the following: degree, number of zeros, number of x-intercept and the number of turning points.

1.



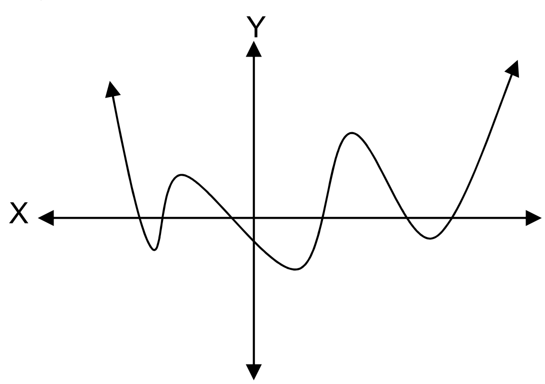
Degree: ____________________
No. of zeros: ________________
No. of x-intercepts: ___________
No. of turning points: __________

2.



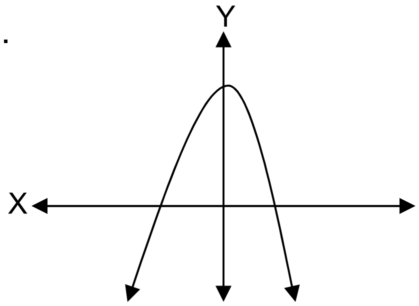
Degree: _____________________
No. of zeros: _________________
No. of x- intercepts: _________ __
No. of turning points: __________

3.



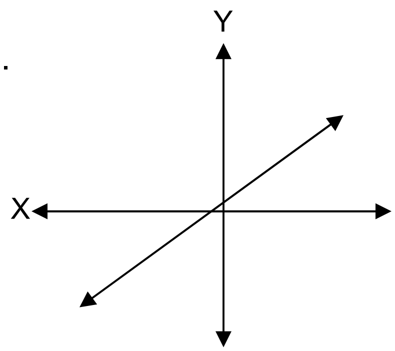
Degree: _____________________
No. of zeros: __________________
No. of x-intercepts: _____________
No. of turning points: ___________

4.



Degree: ________________
No. of zeros: ____________
No. of x-intercepts: ________
No. of turning points: ______

5.



Degree: ______________
No. of zeros: __________
No. of x-intercepts: _____
No. of turning points: ____

C. Draw the graph of the polynomial function f(x) = (x + 3)(2x + 1)(x – 1)(x – 3). Determine the following after drawing the graph:

a. behavior of the graph
b. zeros and the y-intercept of the function.
c. interval where $G(x) \geq 0$.

What you will do

Lesson 1

Graphs of linear and Quadratic Functions

You have learned that the graph of a first-degree polynomial is a line and a second-degree polynomial is a parabola. At this point, recall the very important steps that are undertaken in the construction of a graph. The first step you usually take is to prepare a table of values for x and y that satisfy the given equation. These ordered pairs are points that lie on the curve described by the equation. You then plot the points described by the ordered pairs in the table.

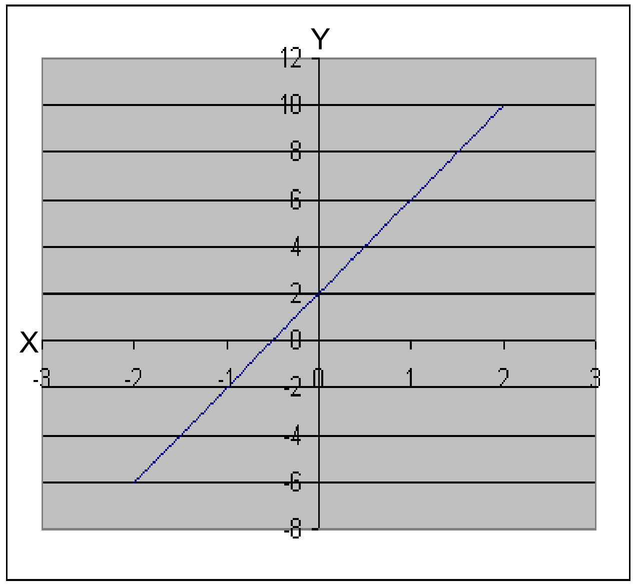
Examples:

1. Graph the line determined by the equation 4x - y = -2. The values of x and y is seen in table 1.1 while the graph is seen in Figure 1.1.

Table 1.1

X	Y
-2	-6
0	2
1	6
2	10

Figure 1.1



Note that the values of x and y are obtained by direct substitution. See these computations.

if x = -2, then
$$y = 4x + 2 = 4(-2) + 2 = -8 + 2 = -6$$

if x = 0, then
$$y = 4x + 2 = 4(0) + 2 = 2$$

if x = 1, then
$$y = 4x + 2 = 4(1) + 2 = 4 + 2 = 6$$

if x = 2, then
$$y = 4x + 2 = 4(2) + 2 = 8 + 10 = 10$$

You have seen that the graph of a linear function is a slanting continuous line. A linear function is of the form f(x) = mx + b. The variable m is the slope of the line, $m \neq 0$, and b is the y-intercept. The zero of the linear function is the value of the independent variable that makes the value of the function 0. It is the x-intercept of the linear function. If the function is y = mx + b, the zero of y is the value of x such that mx + b = 0. The function is increasing when m 〉 0, and decreasing when m 〈 0. You have learned that the domain of a linear function is the set of real numbers and the range is also the set of real numbers.

In the function y = 4x + 2, the degree is one, it has one x-intercept and one zero. It has no turning point since the graph is a slanting line.

Another function that you have learned is the quadratic function whose graph is a parabola that opens upward when a 〉 0 and downward when a 〈 0. A quadratic function is a function that can be described by an equation of the form $f(x) = ax^2 + bx + c = 0$, where $a \neq 0$. The vertex (h, k) of the parabola is the turning point of the graph of a quadratic function.

2. Draw the graph of the quadratic function $y = x^2 - 4x + 4$.

Solution: Find the vertex of the function.

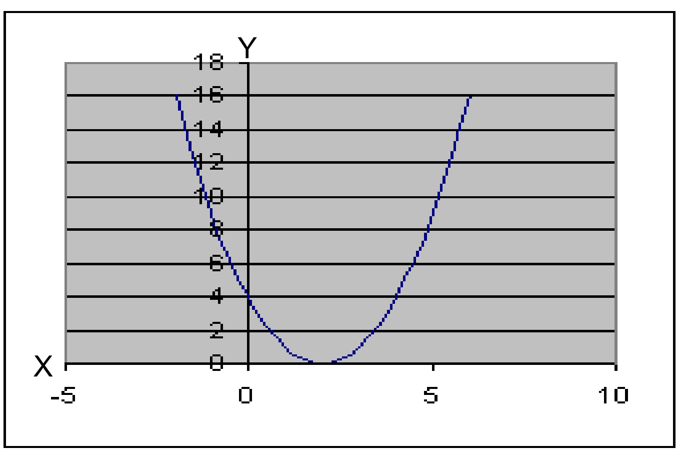
a = 1 b = -4 c = 4

$h = \frac{-b}{2a} = 2$ $k = \frac{4ac - b^2}{4a} = 0$ The vertex is (2, 0).

Table 1.2

X	-2	-1	0	1	2	3	4	5	6
y	16	9	4	1	0	1	4	9	16

Figure 1.2



Try this out

A. Identify at least three points of the given linear function. Then use these points to draw the graph of the function.

1. $y = 5x – 8$
2. $y = -2x + 3$
3. $2x + y – 3 = 0$

B. Determine the vertex, find some representative points then draw the graph

4. $y = x^2 – 7$
5. $y = x^2 - 6x + 9$

Lesson 2A

Graphing Polynomial Functions

In graphing a polynomial function, the technique of finding and plotting as many point as possible will be helpful. But there are theorems concerning roots of polynomial equations that will be of great help to obtain the sketch of the graph of the polynomial function.

The graph of a third-degree polynomials or a cubic polynomial, has both maximum and minimum points. The number of maximum and minimum points is at most one less than the degree of the polynomial. The graph of a polynomial function of degree n has n – 1 turning point.

When you graph higher degree polynomial functions, the rational zero theorem and Rene Descartes' rule can help you find any integral zeros. If there are many possibilities to try, it is also useful to know the upper bound and lower bound for the zeros. An **upper bound** for the real zeros of a polynomial function is a number greater than or equal to the greatest real zero of the function. Similarly, a **lower bound** is a number less than or equal to the least real zero of the function.

Upper and Lower Bound Theorem let a polynomial function be divided by $x - c$.

- If $c \rangle 0$ and all the coefficients in the quotient and remainder are nonnegative, then c is an upper bound of the zeros.
- If $c \langle 0$ and the coefficients in the quotient and remainder alternate in sign, then c is a lower bound of the zeros.

Example:

Find the upper and lower bounds of the zeros of $P(x) = 2x^4 + 5x^3 - 3x^2 - 9x + 1$.

Solution:

It is important to emphasize that c is not restricted to integers. It can be any real number. however, integers are used in illustrative examples for convenience in the computation.

Test c = 1 for upper bound

2	5	-3	-9	1	1
	2	7	4	-5	
2	7	4	-5	-4	

The third entries are not all positive. So, 1 is not an upper bound.

Test c = 2

2	5	-3	-9	1	2
	4	18	30	42	
2	9	15	21	43	

The third entries are all positive, so 2 is an upper bound.

Test c = 3

2	5	-3	-9	1	3
	6	33	90	243	
2	11	30	81	244	

The numbers 3, 4, 5, … are upper bounds. The number 2 is said to be the smallest integral upper bound. This means that there is no zero of P(x) larger than 2.

Test c = -3

2	5	-3	-9	1	-3
	-6	3	0	27	
2	-1	0	-9	28	

The third entries are alternating in signs, so -3 is a lower bound.

If the test for c = -4, -5, -6, … are made, the third entries are alternating. The numbers -4, -5, -6, … are lower bounds. The number -3 is said to be the largest lower bound. This means that there is no zero of P(x) smaller that -3.

Descartes' Rule of Signs

Another information that may be helpful in graphing polynomial functions is the knowledge of the maximum number of positive and negative roots. This information can be provided by the application of Descartes' Rule of signs.

The rule makes use of the number of variations in sign of the coefficients of the polynomial from left to right. The terms must be arranged in descending powers of x. A polynomial is said to have a variation in sign if two consecutive terms have opposite signs.

Descartes' Rule of Signs

Let p(x) = 0 be a polynomial equation with real coefficients, the leading coefficient $a_n \rangle 0$, and is arranged with descending powers of x.

- The number of positive roots of p(x) = 0 is either equal to the number of variations of signs in p(x), or is less than that number by an even counting number.
- The number of negative roots of p(x) = 0 is either equal to the number of variations in signs in p(-x), or is less than that number by an even counting number.

Example:

Determine all possible combinations of number of positive and negative roots of the given polynomial function.

1. $f(x) = 2x^4 + 5x^3 - 2x^2 - 4x + 5$

f(x): + + - - + 2 variations

f(-x): + - - + + 2 variations

No. of positive roots	No. of negative roots	Total number of roots
2	2	4

$f(x) = 0$ has either 2 positive roots and 2 negative roots.

2. $f(x) = x^5 - 5x^4 - 3x^3 + 15x^2 - 4x + 20$

f(x): + - - + - + 4 variations

f(-x): - - + + + + 1 variation

No. of positive roots	No. of negative roots	Total number of roots
4	1	5

$f(x) = 0$ has 4 positive and 1 negative roots.

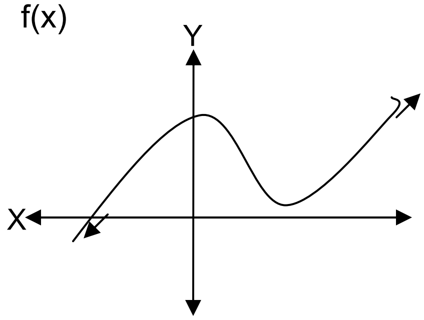
Graph of Odd-degree Polynomials

The extreme left and right parts of the graph of $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ are:

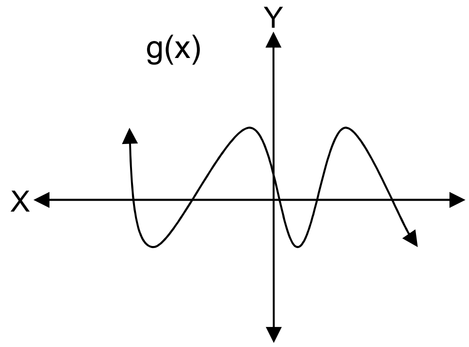
1. increasing; if n is odd and $a_n \rangle 0$.
2. decreasing; if n is odd $a_n \langle 0$.

The domain and range are the set of real numbers.

f(x)



g(x)



f(x) is a polynomial where n = 3 and $a_n > 0$

g(x) is a polynomial where n = 5 and $a_n < 0$

Graph of Even-degree Polynomials

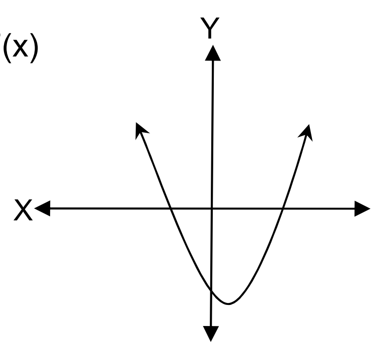
The graph of $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ has

1. decreasing extreme left and increasing extreme right parts, n is even and $a_n > 0$.

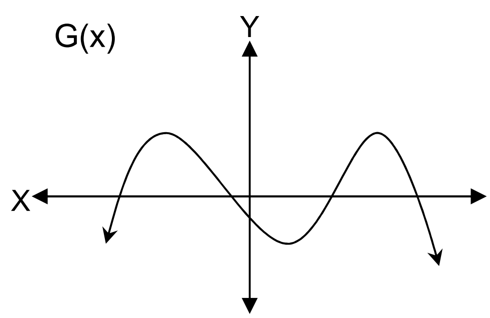
2. increasing extreme left and decreasing extreme right parts, n is even and $a_n < 0$.

The domain is the set of real numbers and the range is the set of nonnegaive numbers.

F(x)



G(x)



F(x) is a polynomial where n = 2 and $a_n > 0$.

G(x) is a polynomial where n = 4 and $a_n < 0$.

Try this out

A. Find the upper and lower bounds of the zeros of each polynomial function.

1. $f(x) = x^3 + 4x^2 + 8x + 5$
2. $p(x) = 6x^4 + x^3 - 56x^2 - 9x + 18$
3. $q(x) = x^5 - 4x^4 - 9x^3 + 3x^2 + 16x - 4$

B. Describe the graph of each polynomial based on its degree and leading coefficients.

4. $f(x) = x^3 + 4x^2 + 8x + 5$
5. $p(x) = 6x^4 + x^3 - 56x^2 - 9x + 18$
6. $q(x) = x^5 - 4x^4 - 9x^3 + 3x^2 + 16x - 4$

C. Use the Descartes' Rule of Signs to determine the possible combinations of roots for each equation.

7. $f(x) = x^3 + 4x^2 + 8x + 5$
8. $p(x) = 6x^4 + x^3 - 56x^2 - 9x + 18$
9. $q(x) = x^5 - 4x^4 - 9x^3 + 3x^2 + 16x - 4$

Lesson 2B

Graphing Polynomial Functions

To graph polynomial function of degree greater than two, evaluate the function to determine ordered pairs, then plot the points with this pairs as coordinates and connect the points to form a smooth curve. The x-coordinates of the points where the graph meets the x-axis are the zeros of the function. The information you can get from the given equation of a polynomial function are the degree and the number of zeros. From the graph, you can determine the number of x-intercepts, y-intercept and number of turning points. The theorems discussed in the previous lesson concerning the roots of polynomial equations will be of great help to obtain the sketch of the graph of the polynomial function.

Examples:

1. $G(x) = x^3 - x^2 - x + 1$

   a. Determine the behavior of the graph
   b. Get the zeros and the y-intercept of the function.
   c. Sketch the graph
   d. Determine the interval where $G(x) \geq 0$.

Solution:

a. The degree of the function is 3 which is an odd, it is an odd-powered function behaving like a line that starts to the right. Since, the leading coefficient is positive. It is increasing.

b. Using synthetic division

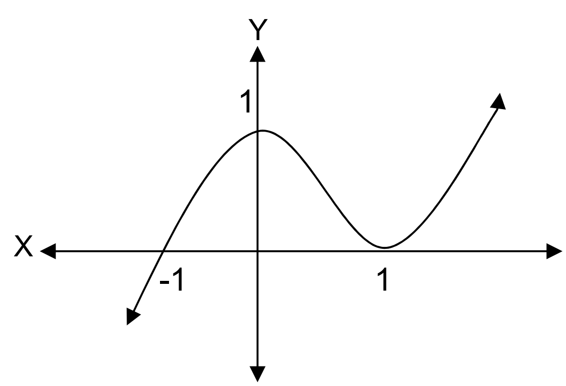
1	-1	-1	1	1
	1	0	-1	
1	0	-1	0	

$$x^2 - 1 = 0$$
$$(x + 1)(x - 1) = 0$$

$$x + 1 = 0 \qquad x - 1 = 0$$
$$x = -1 \qquad x = 1$$

The zeros of the function are -1 and 1 multiplicity 2 and the y-intercept is 1

c.



d. The function G(x) $\geq$ 0 when x $\geq$ -1

e. The graph is tangent to the x-axis at 1, since 1 is a zero of even multiplicity that is 2.

2. $F(x) = x^4 + 2x^3 - 5x^2 - 6x$

a. Determine the behavior of the graph

b. Get the zeros and the y-intercept of the function.

c. Sketch the graph

d. Determine the interval where G(x) $\geq$ 0

Solution:

a. The degree of the function is 4 which is an even, it is an even-powered function behaving like a parabola opening upward.

Extreme bounds:

1	2	-5	-6	0	-4
	-4	8	-12	72	
1	-2	3	-18	72	

-4 is the largest lower bound.

1	2	-5	-6	0	2
	2	8	6	0	
1	4	3	0	0	

2 is the smallest integral upper bound and a zero.

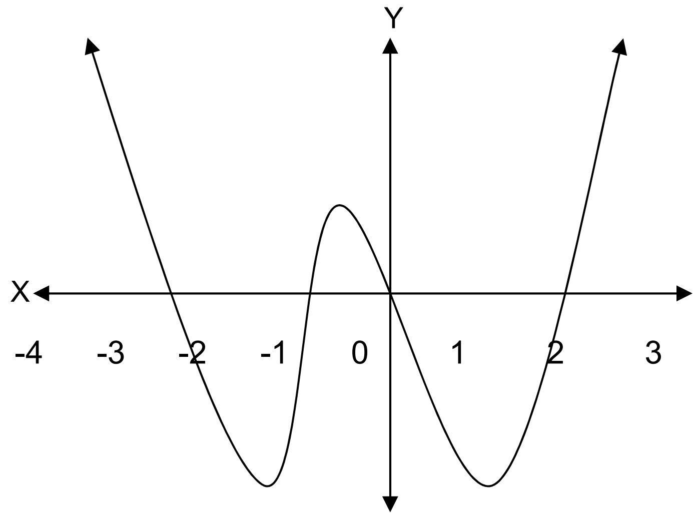
b. Location of zeros

x	-4	-3	-2	-1	0	1	2
F(x)	72	0	-8	0	0	-8	0

The table reveals the following information:

1. The zeros of F(x) are -3, -1, 0 and 2.
2. y-intercept is 0.

c. Sketch of the graph.



Since n = 4, the function F(X) has three turning points.

d. Since n = 4 and a_n = 1, the extreme left part is decreasing and the extreme right part is increasing. The function F(x) 〉 0 when x 〈 -3, when -1 〈 x 〈 0 or x 〉 2.

3. $p(x) = x^3 - 4x^2 - 4x + 16$

a. Determine the behavior of the graph
b. Get the zeros and the y-intercept of the function.
c. Sketch the graph
d. Determine the interval where $p(x) \geq 0$.

Solution:

a. The degree of the function is 3 which is an odd, it is an odd-powered function behaving like a line that starts from the right. Since the leading coefficient is positive, it is increasing.

b. Possible combination zeros: ±1, ±2, ±4, ±8, ±16

Extreme bounds:

1	-4	-4	16	5
	5	5	5	
1	1	1	21	

5 is the least integral upper bound.

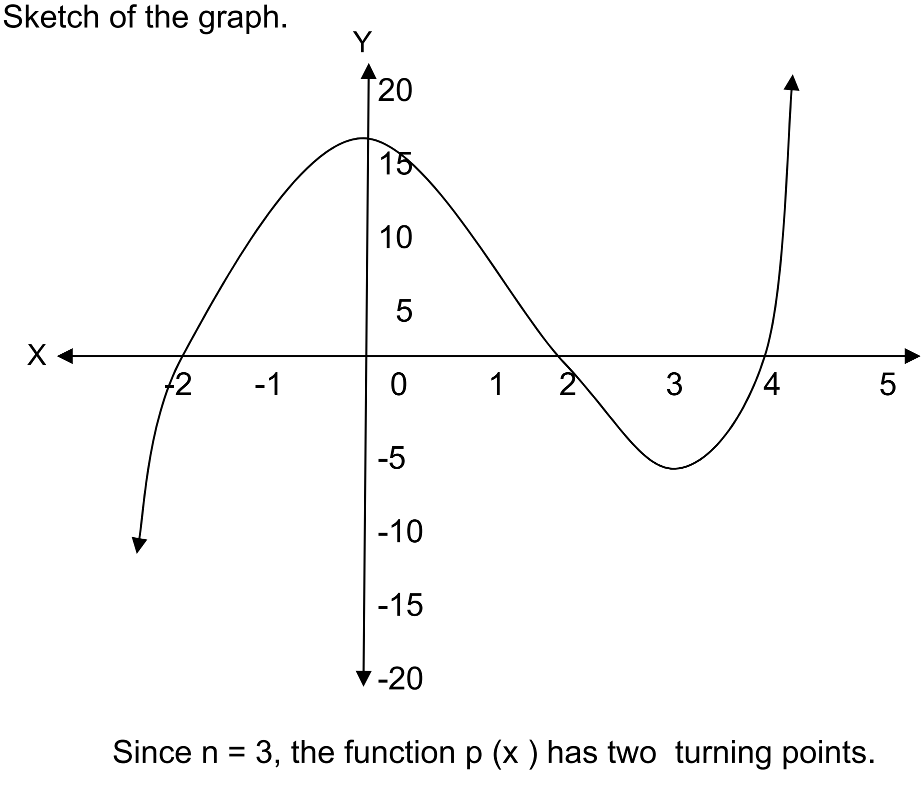
1	-4	-4	16	-2
	-2	12	-16	
1	-6	8	0	

-2 is the greatest integral lower bound and a zero.

b. Location of zeros

X	-2	-1	0	1	2	3	4	5
Y	0	15	16	9	0	-5	0	21

c. Sketch of the graph.



Since n = 3, the function p (x) has two turning points.

The table reveals the following information:

1. The zeros of p(x) are -2, 2 and 4.
2. y-intercept is 16.

d. The function p(x) $\geq$ 0 if 2 $\geq$ x < 2 or x > 4.

4. G(x) = $-x^5 + 15x^3 + 10x^2 - 24x$

a. Determine the behavior of the graph

b. Get the zeros and the y-intercept of the function.

c. Sketch the graph

Solution:

a. The degree of the function is 5 which is an odd, it is an odd-powered function behaving like a line that starts from the left since the leading coefficient is negative -1. The extreme left and right parts are decreasing.

b. Possible zeros: ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24

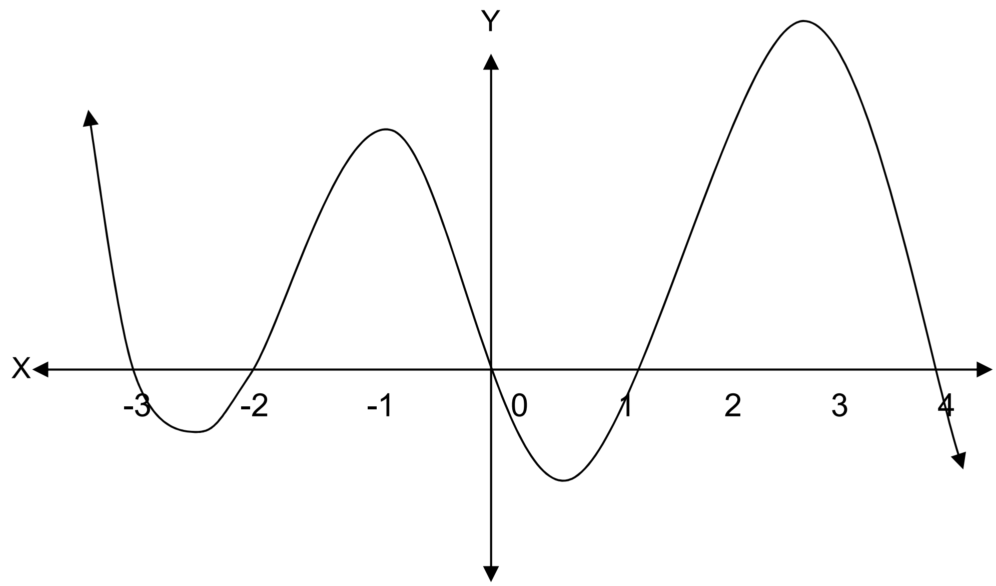
The theorems on bounds does not apply for G(x) since the leading coefficient is negative.

x	-3	-2	-1	0	1	2	3	4
y	0	0	20	0	0	80	180	0

The table reveals that the following information:

1. The 5 zeros are -3, -2, 0, 1 and 4
2. The y intercept is 0.

c. Sketch the graph.



5. $f(x) = x^3 - 3x^2 - 5x + 12$ using a graphing calculator and approximate its real zeros to the nearest half unit.

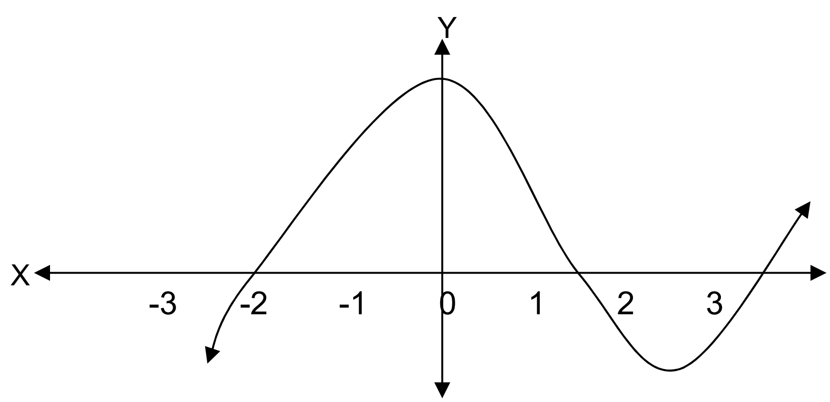
Steps:

To draw the graph:

1. Use the GRAPH Mode input in the function.

2. Press F6 to draw the graph.

From your calculator screen you will see the sketch of the graph like this.



To determine the zeros of $f(x) = x^3 - 3x^2 - 5x + 12$

1. Press F5 (G-Solv)
2. Press F1 (Root) to display one root.
3. Press the $\rightarrow$ arrow key (3x) for other roots.

- Press F5 (G – Solv) to use othe features such as finding the y-intercept and roots.
- Observe the behavior of the graphs when it is odd or even.

Note from the graph that the real zeros of $f(x) = x^3 - 3x^2 - 5x + 12$, to the nearest half unit, are -2.0, 1.5 and 3.5.

Try this out

Graph each of the following polynomials.

a. Determine the behavior of the graph

b. Get the zeros and the y-intercept of the function.

c. Sketch the graph

1. $f(x) = x^3 – x^2 – 4x + 4$
2. $F(x) = 2x^4 + 9x^3 + 11x^2 - 4$
3. $H(x) = -x^4 + 2x^3 + 3x^2 - 2x + 6$
4. Graph the function and approximate its real zeros to the nearest tenth using graphing calculator:

$$y = 2x^4 + 5x^3 - 3x^2 - 9x + 1$$

Let's summarize

To graph polynomial function of degree greater than two, evaluate the function to determine ordered pairs, then plot the points with this pairs as coordinates and connect the points to form a smooth curve. The x-coordinates of the points where the graph meets the x-axis are the zeros of the function. The information you can get from the given equation of a polynomial function are the degree and the number of zeros. From the graph, you can determine the number of x-intercepts, y-intercept and number of turning points.

Summary of the characteristics of the graph of polynomial function:

a_n	n	Properties of the graph	Illustration
+	Even	Comes down from the left, goes up to the right	Figure 1
+	Odd	Comes up from the left, goes up to the right	Figure 2
-	Even	Comes up from the left, goes down to the right	Figure 3
-	Odd	Comes down from the left, goes down to the right	Figure 4

Figure 1

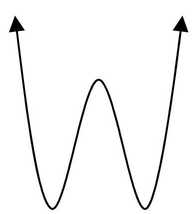


Figure 2

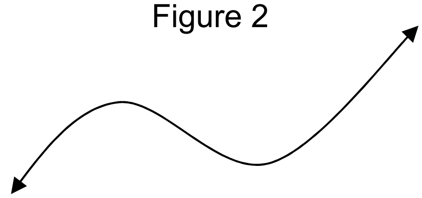


Figure 3

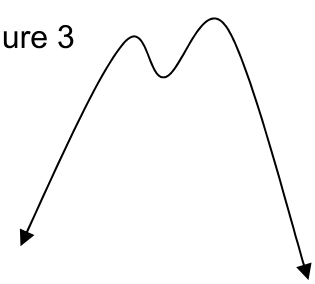
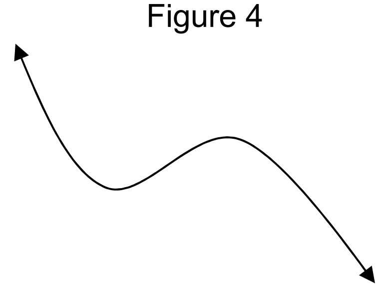


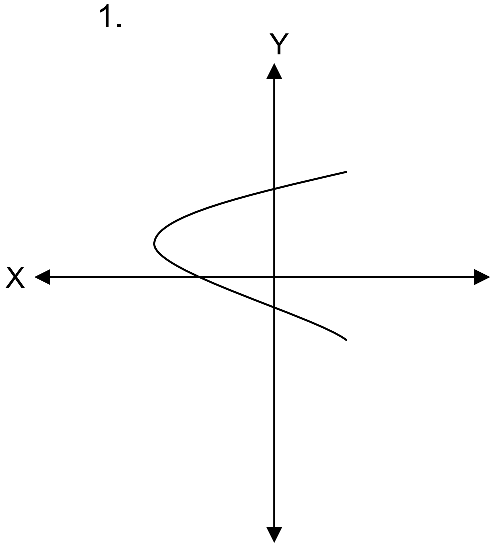
Figure 4



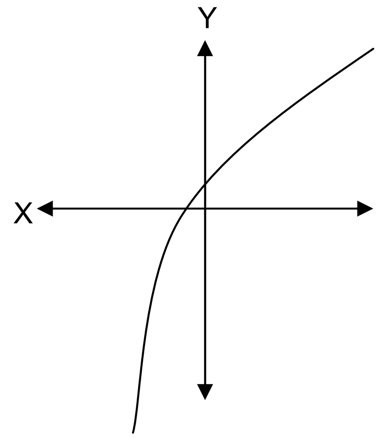
What have you learned

A. Indicate whether the graph represents a polynomial function.

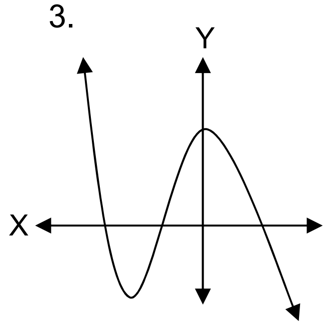
1.



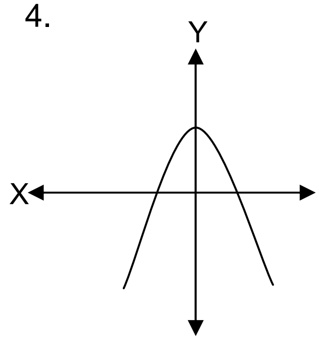
2.



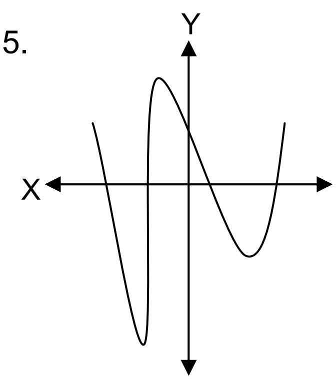
3.



4.



5.



B. Complete the table below:

Functions	Degree	Number of zeros	No. of x-intercepts	No. of turning points
1. $P(x) = (x-1)(x+2)(x-1)$				
2. $y = x(x-2)$				
3. $g(x) = x^4 - x^3 - 7x^2 + x + 6$				
4. $P(x) = 2x^5 + 5x^4 - 2x^3 - 7x^2 - 4x - 12$				

C. For each of the following polynomials,

a. determine behavior of the graph.

b. get the zeros and the y-intercept of the function.

c. sketch the graph

d. interval where $G(x) \geq 0$

1. $P(x) = x^4 + x^3 + 4x^2 + 6x - 12$

2. $P(x) = -x^5 + 15x^3 + 10x^2 - 24x$

Answer Key

How much do you know

A.1. linear 2. quadratic 3. quadratic 4. polynomial 5. polynomial

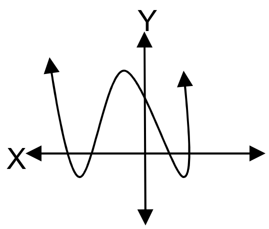
B.

	degree	No. of zeros	No. of x-intercept	No. of turning points
1	2	2	2	1
2	3	3	3	2
3	6	6	6	5
4	2	2	2	1
5	1	1	1	0

C. a. behaves like a parabola that opens upward

b. Zeros are -3, $\frac{-1}{2}$, 1 and 3, y-intercept is 0

c. $f(x) \geq 0$ when $x \langle -3$, when $\frac{-1}{2} \langle x \langle 1$ or when $x \rangle 3$



Lesson 1

1.

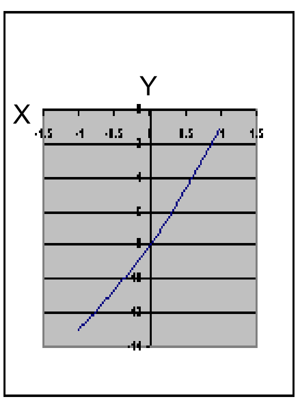
X	-1	0	1
y	-13	-8	-3

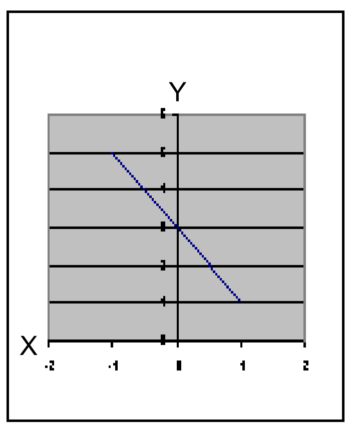
2.

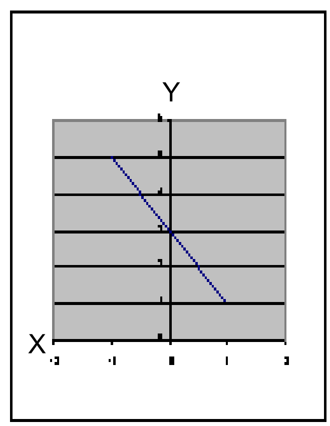
X	-1	0	1
Y	5	3	1

3.

X	-1	0	1
y	5	3	1

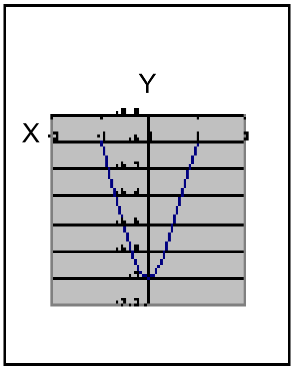






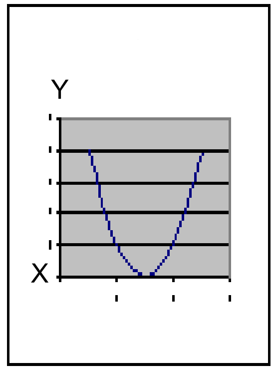
4. (0, -7)

X	-1	0	1
Y	-6	-7	-6



5. (3, 0)

X	1	2	3	4	5
Y	4	1	0	1	4



Lesson 2a

A. 1. upper bound: -1
Lower bound: -4

2. upper bound: 4
lower bound: -4

3. upper bound: 6
lower bound: -2

B. 4. n = 3 and a_n = 1. The extreme left and right parts of the function are increasing.

5. n = 4 and a_n = 6. The extreme left part is decreasing and the extreme right is increasing.

6. n = 5 and a_n = 1. The extreme left and right parts are increasing.

C. 7.

+ zeros	- zeros	Imaginary zeros
0	3	0

8.

+ zeros	- zeros	Imaginary zeros
2	2	0

9.

+ zeros	- zeros	Imaginary zeros
3	2	0

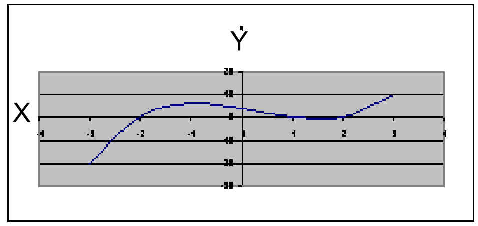
Lesson 2B:

1. a. The extreme left and right parts of the function are increasing.

   b. Zeros are 1, -2 and 3 and the y-intercept is 4

   c.

x	-3	-2	-1	0	1	2	3
y	-20	0	6	4	0	0	10



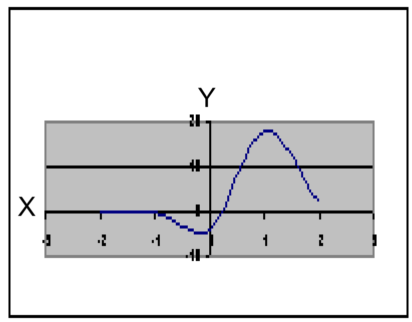
   d. f(x) ≥0 for all x〉-2

2. . a. The graph comes from the left and goes up to the right..

   b. Zeros are -1, -2, $\frac{1}{2}$, and 2. The y-intercept is -4.

   c.

x	-2	-1	0	1	2
y	0	0	-4	18	2

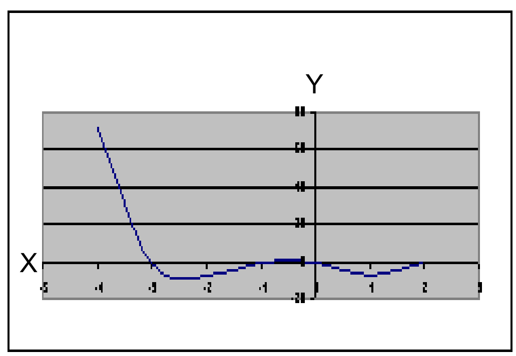


3. a. The extreme left part is decreasing and the extreme right part is increasing.

   b. Zeros are -3, -1, 0, and 2. The y-intercept is 0.

   c.

x	-4	-3	-2	-1	0	1	2
y	72	0	-8	0	0	-8	0



What have you learned

A. 1. not polynomial   nos. 2 – 5 are polynomials

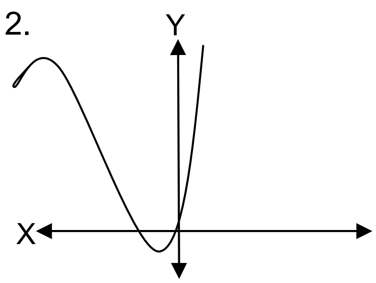
B.

Functions	Degree	Number of zeros	No. of x-intercepts	No. of turning points
1. $P(x) = (x-1)(x+2)(x-1)$	3	3	3	2
2. $y = x(x-2)$	2	2	2	1
3. $g(x) = x^4 - x^3 - 7x^2 + x + 6$	4	4	4	3
4. $P(x) = 2x^5 + 5x^4 - 2x^3 - 7x^2 - 4x - 12$	5	5	5	4

C.

1. a. The graph comes from the left and goes up to the right.

   b. The real zeros are -2 and 1. The y-intercept is -12.



   c.

x	-4	-3	-2	-1	0	1
y	220	72	0	-14	-12	0

   d. $P(x) \geq 0$ for all $x > 1$ or $x < -2$

2 a. The extreme left part is decreasing and the extreme right part is increasing.

   b. Zeros are -3, -2, 0, 1 and 4. The y-intercept is 0

   c.

x	-3	-2	1	0	1	2	3	4
y	0	0	20	0	0	80	180	0

   d. $f(x) \geq 0$ for $x > 1$