Module 2
Polynomial Functions

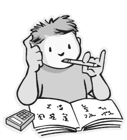
What this module is about

This module is about finding the zeros of polynomial functions of degree greater than 2. In module 1, the factor theorem was introduced to you by simply stating, if zero is obtained as a remainder when c is substituted to the polynomial P(x), then the polynomial x – c is factor of P(x). This time, you will learn different methods of finding the zeros of polynomial functions.

What you are expected to learn

This module is designed for you to find the zeros of polynomial functions of degree greater than 2 by:

a. Factor Theorem
b. factoring
c. synthetic division
d. depressed equations



How much do you know

1. How many zeros do the polynomial function $f(x) = 2x^5 – 3x^4 – x^3 + 2x^2 + x – 3$ have?
2. How many roots do the polynomial equation $6x^4 + 11x^3 + 8x^2 – 6x – 4 = 0$ have?
3. Determine the zeros of the polynomial function $F(x) = x(x – 3)^2(x + 1)(2x – 3)$.
4. What are the possible rational zeros of $p(x) = x^4 – 9x^3 + 23x^2 – 15$?
5. What are the possible rational roots $3x^5 – x^4 + 6x^3 – 2x^2 + 8x – 5 = 0$?
6. Find all the zeros of $h(x) = x^3 – 10x^2 + 32x – 32$.
7. Solve the polynomial equation $x^4 – 6x^3 – 9x^2 + 14x = 0$ using synthetic division.
8. Find all zeros of $g(x) = x^3 – 2x^2 – x + 2$ using depressed equations.

9. One of the roots of $x^3 - 12x^2 - 8x + 96 = 0$ is $2\sqrt{2}$. What are the other roots?
10. One of the zeros of $p(x) = 2x^4 - x^3 + 25x^2 - 13x - 13$ is $-i\sqrt{13}$. Find the other zeros.

What you will do

Lesson 1

Number of Roots Theorem

The Fundamental Theorem of Algebra which is attributed to Karl Freidrich Gauss of Germany states that "Every polynomial equation in one variable has at least one root, real or imaginary". The next theorem tells us of the exact number of roots of polynomial equation of degree n:

"Every polynomial equation of a degree $n \geq 1$ has exactly n roots."

Examples:

Determine the number of roots of each polynomial equation.

1. $3x^7 + 8x^5 - 4x - 1 = 0$

   $3x^7 + 8x^5 - 4x - 1 = 0$ is of the seventh degree. Hence it has 7 roots.

2. $(x - 1)(2x + 1)^3(2x - 5)^2 = 0$

   $(x - 1)(2x + 1)^3(2x - 5)^2 = 0$ is of the sixth degree. Hence it has 6 roots.

3. $x(x - a)^m(x + b)^n = 0$

   $x(x - a)^m(x + b)^n = 0$ is of the $(1 + m + n)$th degree. Hence it has $1 + m + n$ roots.

Try this out

Determine the number of roots of each polynomial equation.

Set A
1. $x^5 + 2x^3 - x - 3 = 0$
2. $-x^7 + 2x^6 - 4x^5 - x^2 + 2x - 1 = 0$

3. $2 + x^2 - 3x^4 - x^6 - x^8 - 2x^{10} = 0$
4. $(x - 5)(x + 2)^3(2x - 1)^2 = 0$
5. $x^2(x + 1)(x - 3)^4 = 0$

Set B

1. $8x^3 - 9x + 1 = 0$
2. $-4x^7 - 6x^6 + x^2 - 2x + 5 = 0$
3. $(x - 2)(x + 9)^3x^4 = 0$
4. $3x^3(x + 8)^2(x^2 - 4) = 0$
5. $(x^2 - 1)(x^3 + 1) = 0$

Set C

1. $x\left(x + \frac{1}{2}\right)^3 (x^2 + 2)(x + 1) = 0$
2. $x^4\left(\frac{9}{4}x - \frac{3}{2}\right)(2x - 3) = 0$
3. $(x^2 - 2x + 1)(x^3 - 1)(x^2 + x - 6) = 0$
4. $x(x^2 + 2)^2(x^2 + 2x - 1)^2 = 0$
5. $-9x^4(8 - x^3)^2 = 0$

Lesson 2

Determining the Zeros of Polynomial Functions in Factored Form

Recall that a zero of p(x) is the value of x that will make the function 0. The zeros of a polynomial function in factored form are determined by equating each factor to 0 and solving for x.

Examples:

Determine the zeros of each polynomial function.

1. $f(x) = x(x + 3)(x - 2)$

   Equate each factor to 0 and solve for x.

   $x = 0$
   $x + 3 = 0, x = -3$
   $x - 2 = 0, x = 2$

   Therefore, the zeros of f(x) are 0, -3, and 2.

2. $y = (5x - 2)(2x + 1)(-3x - 4)$

   Equate each factor to 0 and solve for x.

   $5x - 2 = 0,\ x = \frac{2}{5}$

   $2x + 1 = 0,\ x = -\frac{1}{2}$

   $-3x - 4 = 0,\ x = -\frac{4}{3}$

   Therefore, the zeros of f(x) are $\frac{2}{5}$, $-\frac{1}{2}$, and $-\frac{4}{3}$.

3. $g(x) = (x + 4)^3(x - 3)(2x - 1)^2$

g(x) has 3 factors of (x + 4), 1 factor of (x – 3) and 2 factors of (2x – 1). Thus, the zeros of g(x) are: -4 of multiplicity 3; 3 of multiplicity 1; and $\frac{1}{2}$ of multiplicity 2.

4. $h(x) = (x^2 - 4)(x^2 - 3x - 28)$

   Equate each factor to 0 and solve for x.

   $x^2 - 4 = (x + 2)(x - 2) = 0$, x = -2 and x = 2
   $x^2 - 3x - 28 = (x - 7)(x + 4) = 0$, x = 7 and x = -4

   Thus, the zeros of h(x) are -2, 2, 7, and -4.

Try this out

Determine the zeros of each polynomial function.

Set A
1. $f(x) = x(x + 4)(x - 2)$
2. $g(x) = -x(x + 7)(x - 1)$
3. $h(x) = (4x - 5)(2x + 3)(x - 3)$
4. $F(x) = x(4 - 3x)(1 - x)$
5. $G(x) = x(x + 3)(3x + 1)$

Set B
1. $F(x) = (x - 8)^5(x + 2)^3$
2. $G(x) = (5x + 1)^6(2x - 7)^4$
3. $H(x) = x(5 - x)(2 - 3x)^2$
4. $f(x) = x^2(2x - 3)(x + 4)^3(3x - 7)$
5. $g(x) = -7x^3(x - 4)(5x + 2)^4(x - 1)^2$

Set C

1. $h(x) = x(x + 3)(2x - 9)(3x + 1)$
2. $k(x) = (x^2 - 1)(4x^2 - 4x + 1)$
3. $p(x) = (x^2 + 7x + 10)(9x^2 - 12x)$
4. $y = (2x^3 + 3x^2 - 5x)(12x^2 + 34x + 14)$
5. $y = (-2x^2 - x + 3)(12x^2 + 23x + 5)$

Lesson 3

The Rational Roots Theorem

The next theorem specifies a finite set of rational numbers where the roots of a polynomial equation can be chosen.

"If a rational number $\frac{L}{F}$ in lowest terms is a root of the polynomial equation $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 = 0$, where a_n, a_{n-1}, a_{n-2}, …, a_2, a_1, a_0 are integers, then L is a factor of a_0 and F is a factor of a_n."

The Rational Roots Theorem states that "Any rational root of the polynomial equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 = 0$, where a_{n-1}, a_{n-2}, … , a_2, a_1, a_0 are integers, is an integer and is a factor of a_0."

Examples:

List all possible zeros of the given polynomial function.

1. $f(x) = x^3 - 6x^2 + 11x - 6$

   Since the coefficient of the highest degree term is 1, the possible rational zeros of f(x) are the factors of the constant term -6. That is, the possible rational zeros are ±1, ±2, ±3, and ±6.

2. $g(x) = x^4 - x^3 - 11x^2 + 9x + 18$

   Since the coefficient of the highest degree term is 1, the possible rational zeros of g(x) are the factors of the constant term 18. That is, the possible rational zeros are ±1, ±2, ±3, ±6, ±9 and ±18.

3. $h(x) = 2x^4 + 9x^3 + 11x^2 - 4$

   If we let L = the factors of -4: ±1, ±2, ±4,
   and F = the factors of 2: ±1, ±2.

Then $\frac{L}{F}$ are $\pm\frac{1}{1}=\pm 1$, $\pm\frac{1}{2}$, $\pm\frac{2}{1}=\pm 2$, $\pm\frac{4}{1}=\pm 4$, and $\pm\frac{4}{2}=\pm 2$

or $\frac{L}{F}$ = ±1, ±2, ±4, $\pm\frac{1}{2}$.

4. $p(x) = 8x^4 + 32x^3 + x + 4$

   L = the factors of 4: ±1, ±2, ±4
   F = the factors of 8: ±1, ±2, ±1, ±4, ±8

   The possible rational zeros $\frac{L}{F}$ are $\pm\frac{1}{1}=\pm 1$, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$, $\pm\frac{2}{1}=\pm 2$, $\pm\frac{2}{2}=\pm 1$, $\pm\frac{2}{4}=\pm\frac{1}{2}$, $\pm\frac{2}{8}=\pm\frac{1}{4}$, $\pm\frac{4}{1}=\pm 4$, $\pm\frac{4}{2}=\pm 2$, $\pm\frac{4}{4}=\pm 1$ and $\pm\frac{4}{8}=\pm\frac{1}{2}$

   or $\frac{L}{F}$ = ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, $\pm\frac{1}{8}$, ±2, and ±4.

Try this out

List all possible zeros of the given polynomial function.

Set A
1. $f(x) = x^3 – 4x^2 – 2x + 5$
2. $g(x) = x^3 – 6x^2 + 2x – 6$
3. $h(x) = x^3 – x^2 – 5x – 3$
4. $p(x) = x^4 + 2x^3 – 8x – 16$
5. $y = 2x^3 + 17x^2 + 23x – 42$

Set B
1. $f(x) = x^5 + x^4 – x – 1$
2. $g(x) = x^4 + 32$
3. $h(x) = 2x^3 + 3x^2 – 8x + 3$
4. $p(x) = 3x^3 + 13x^2 + 9x + 20$
5. $y = 4x^4 + 16x^3 + 9x^2 – 32$

Set C
1. $f(x) = x^3 – 7x^2 – 15$
2. $g(x) = 2x^3 – x^2 – 4x + 2$
3. $h(x) = 3x^3 – 2x^2 + 3x – 2$
4. $p(x) = 3x^3 + 4x^2 + 12x + 16$
5. $y = 6x^4 + x^3 – 13x^2 – 2x + 2$

Lesson 4

Determining the Zeros of Polynomial Functions Using the Factor Theorem

The Factor Theorem states that "If p(c) = 0, then x – c is a factor of p(x)." This implies that c is a zero of p(x).

To determine the rational zeros of a polynomial function from the list of all possible rational zeros using the Factor Theorem, evaluate the polynomial function using these possible zeros one at a time. If a zero was obtained after evaluating a particular rational zero, then you can say that that number is a zero of the polynomial

Examples:

Determine the rational zeros of the given polynomial function using the Factor Theorem.

1. $f(x) = x^3 + 6x^2 + 11x + 6$

   There are 3 zeros, real or imaginary. According to the Rational Roots Theorem, the possible rational zeros are ±1, ±2, ±3, and ±6.

   If x = -1, then $f(x) = x^3 + 6x^2 + 11x + 6$ becomes

   $$\begin{aligned} f(-1) &= (-1)^3 + 6(-1)^2 + 11(-1) + 6 \\ &= -1 + 6 - 11 + 6 \\ &= 0 \end{aligned}$$

   -1 is a zero of f(x).

   If x = 1, then $f(x) = x^3 + 6x^2 + 11x + 6$ becomes

   $$\begin{aligned} f(1) &= (1)^3 + 6(1)^2 + 11(1) + 6 \\ &= 1 + 6 + 11 + 6 \\ &= 24 \end{aligned}$$

   1 is not a zero of f(x).

   If x = -2, then $f(x) = x^3 + 6x^2 + 11x + 6$ becomes

   $$\begin{aligned} f(-2) &= (-2)^3 + 6(-2)^2 + 11(-2) + 6 \\ &= -8 + 24 - 22 + 6 \\ &= 0 \end{aligned}$$

   -2 is a zero of f(x).

If $x = 2$, then $f(x) = x^3 + 6x^2 + 11x + 6$ becomes

$$f(2) = (2)^3 + 6(2)^2 + 11(2) + 6$$
$$= 8 + 24 + 22 + 6$$
$$= 60$$

-1 is not a zero of f(x).

If $x = -3$, then $f(x) = x^3 + 6x^2 + 11x + 6$ becomes

$$f(-3) = (-3)^3 + 6(-3)^2 + 11(-3) + 6$$
$$= -27 + 54 – 33 + 6$$
$$= 0$$

-3 is a zero of f(x).

f(x) has only 3 zeros and we have already found 3. Thus the zeros are -1, -2 and -3.

2. $g(x) = x^4 – x^3 – 11x^2 + 9x + 18$

There are 4 zeros, real or imaginary. According to the Rational Roots Theorem, the possible rational zeros are ±1, ±2, ±3, ±6, ±9 and ±18.

If $x = -1$, then $g(x) = x^4 – x^3 – 11x^2 + 9x + 18$ becomes

$$g(-1) = (-1)^4 – (-1)^3 – 11(-1)^2 + 9(-1) + 18$$
$$= 1 + 1 – 11 – 9 + 18$$
$$= 0$$

-1 is a zero of g(x).

If $x = 1$, then $g(x) = x^4 – x^3 – 11x^2 + 9x + 18$ becomes

$$g(1) = (1)^4 – (1)^3 – 11(1)^2 + 9(1) + 18$$
$$= 1 – 1 – 11 + 9 + 18$$
$$= 16$$

1 is not a zero of g(x).

If $x = -2$, then $g(x) = x^4 – x^3 – 11x^2 + 9x + 18$ becomes

$$g(-2) = (-2)^4 – (-2)^3 – 11(-2)^2 + 9(-2) + 18$$
$$= 16 + 8 – 22 – 18 + 18$$
$$= 2$$

-1 is not a zero of g(x).

If $x = 2$, then $g(x) = x^4 – x^3 – 11x^2 + 9x + 18$ becomes

$$g(2) = (2)^4 – (2)^3 – 11(2)^2 + 9(2) + 18$$
$$= 16 – 8 – 44 + 18 + 18$$
$$= 0$$

2 is a zero of g(x).

If x = -3, then g(x) = $x^4 - x^3 - 11x^2 + 9x + 18$ becomes

$$g(-3) = (-3)^4 - (-3)^3 - 11(-3)^2 + 9(-3) + 18$$
$$= 81 + 27 - 99 - 27 + 18$$
$$= 0$$

-3 is a zero of g(x).

If x = 3, then g(x) = $x^4 - x^3 - 11x^2 + 9x + 18$ becomes

$$g(3) = (3)^4 - (3)^3 - 11(3)^2 + 9(3) + 18$$
$$= 81 - 27 - 99 + 27 + 18$$
$$= 0$$

3 is a zero of g(x).

g(x) has only 4 zeros and we have already found 4. Thus the zeros are -1, 2 -3 and 3.

3. h(x) = $2x^4 + 3x^3 + 3x - 2$

There are 4 zeros, real or imaginary. The possible rational zeros are $\pm\frac{1}{2}$, ±1 and ±2.

If x = $-\frac{1}{2}$, then h(x) = $2x^4 + 3x^3 + 3x - 2$ becomes

$$h\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^4 + 3\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right) - 2$$
$$= \frac{1}{8} - \frac{3}{8} - \frac{3}{2} - 2$$
$$= -\frac{15}{4}$$

$-\frac{1}{2}$ is not a zero of h(x).

If x = $\frac{1}{2}$, then h(x) = $2x^4 + 3x^3 + 3x - 2$ becomes

$$h\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 + 3\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right) - 2$$
$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{2} - 2$$
$$= 0$$

$\frac{1}{2}$ is a zero of h(x).

If x = -1, then $h(x) = 2x^4 + 3x^3 + 3x – 2$ becomes

$$
\begin{aligned}
h(-1) &= 2(-1)^4 + 3(-1)^3 + 3(-1) – 2 \\
&= 2 – 3 – 3 – 2 \\
&= -6
\end{aligned}
$$

-1 is not a zero of h(x).

If x = 1, then $h(x) = 2x^4 + 3x^3 + 3x – 2$ becomes

$$
\begin{aligned}
h(1) &= 2(1)^4 + 3(1)^3 + 3(1) – 2 \\
&= 2 + 3 + 3 – 2 \\
&= 6
\end{aligned}
$$

1 is not a zero of h(x).

If x = -2, then $h(x) = 2x^4 + 3x^3 + 3x – 2$ becomes

$$
\begin{aligned}
h(-2) &= 2(-2)^4 + 3(-2)^3 + 3(-2) – 2 \\
&= 32 – 24 – 6 – 2 \\
&= 0
\end{aligned}
$$

-1 is a zero of h(x).

If x = 2, then $h(x) = 2x^4 + 3x^3 + 3x – 2$ becomes

$$
\begin{aligned}
h(2) &= 2(2)^4 + 3(2)^3 + 3(2) – 2 \\
&= 32 + 24 + 6 – 2 \\
&= 60
\end{aligned}
$$

2 is not a zero of h(x).

g(x) has 4 zeros and we have already used all possible rational zeros. We found only 2 rational zeros. This indicates that there are only 2 rational zeros. The other 2 zeros are not rational; they may be irrational or imaginary. (Irrational and imaginary zeros will be discussed in the succeeding lessons.) Thus the only rational zeros are $\frac{1}{2}$ and -2.

Try this out

Determine the rational zeros of the given polynomial function using the Factor Theorem.

Set A

1. $f(x) = x^3 + 2x^2 – 5x – 6$
2. $g(x) = x^3 + 4x^2 + x – 6$
3. $h(x) = x^3 + 3x^2 – 4x – 12$
4. $p(x) = x^3 – x^2 – 10x – 8$
5. $y = x^3 + x^2 – x – 1$

Set B

1. $f(x) = x^3 – 4x^2 + x + 6$
2. $g(x) = x^3 – 5x^2 – 2x + 24$
3. $h(x) = x^3 – 3x^2 – 4x + 12$
4. $p(x) = x^3 – 6x^2 + 5x + 12$
5. $y = x^3 – 5x^2 – x + 5$

Set C

1. $f(x) = x^3 – x^2 – x + 1$
2. $g(x) = 2x^3 – 11x^2 – 8x + 12$
3. $h(x) = 3x^3 – 2x^2 - 27x + 18$
4. $p(x) = 4x^4 – 5x^2 + 1$
5. $y = 2x^4 + 9x^3 + 6x^2 – 5x – 6$

Lesson 5

Determining the Zeros of Polynomial Functions by Factoring

The zeros of a polynomial function can be determined easily if the polynomial is in factored form. But the problem arises when the polynomial is expressed otherwise. The polynomial must be factored (if it is factorable) using techniques learned in elementary algebra.

Examples:

Determine the rational zeros of the given polynomial function by factoring.

1. $f(x) = x^3 – 3x^2 – 6x + 8$

$$\begin{aligned} f(x) &= x^3 – x^2 – 2x^2 – 6x + 8 \quad \text{Using factoring by grouping.} \\ &= (x^3 – x^2) – (2x^2 + 6x – 8) \\ &= x^2(x – 1) – 2(x^2 + 3x – 4) \\ &= x^2(x – 1) – 2(x + 4)(x – 1) \\ &= (x – 1)[x^2 – 2(x + 4)] \\ &= (x – 1)(x^2 – 2x – 8) \\ &= (x – 1)(x + 2)(x – 4) \end{aligned}$$

$x - 1 = 0$ $\quad$ $x + 2 = 0$ $\quad$ $x – 4 = 0$ $\quad$ Equate factors to zero

$x = 1$ $\quad$ $x = -2$ $\quad$ $x = 4$

Hence, the zeros of f(x) are 1, -2, and 4

2. $g(x) = x^3 + 2x^2 – 5x – 6$

$g(x) = x^3 + (x^2 + x^2) – 5x – 6$ Using factoring by grouping.
$= (x^3 + x^2) + (x^2 – 5x – 6)$
$= x^2(x + 1) + (x + 1)(x – 6)$
$= (x + 1)[x^2 + (x – 6)]$
$= (x + 1)(x^2 + x – 6)$
$= (x + 1)(x + 3)(x – 2)$

$x + 1 = 0$ $x + 3 = 0$ $x – 2 = 0$ Equate factors to zero
$x = -1$ $x = -3$ $x = 2$

Hence, the zeros of g(x) are -1, -3, and 2.

3. $h(x) = x^4 + 4x^3 + x^2 – 6x$

$h(x) = x(x^3 + 4x^2 + x – 6)$ Using common monomial factoring
$= x(x^3 + 2x^2 + 2x^2 + x – 6)$ Factoring by grouping
$= x[(x^3 + 2x^2) + (2x^2 + x – 6)]$
$= x[x^2(x + 2) + (x + 2)(2x – 3)]$
$= x\{(x + 2)[x^2 + (2x – 3)]\}$
$= x[(x + 2)(x^2 + 2x – 3)]$
$= x(x + 2)(x + 3)(x – 1)$

$x = 0$ $x + 2 = 0$ $x + 3 = 0$ $x – 1 = 0$ Equate factors to 0.
$x = -2$ $x = 1$ $x = 1$

Hence, the zeros of h(x) are 0, -2, -3, and 1.

Try this out

Determine the rational zeros of the given polynomial function by factoring.

Set A
1. $f(x) = x^3 + 3x^2 – 4x – 12$
2. $g(x) = x^3 + 2x^2 – 5x – 6$
3. $h(x) = x^3 – x^2 – 10x – 8$
4. $p(x) = x^3 + 4x^2 + x – 6$
5. $y = x^3 + x^2 – x – 1$

Set B
1. $f(x) = x^3 – 6x^2 + 5x + 12$
2. $g(x)= x^3 – 3x^2 – 4x + 12$
3. $h(x) = x^3 – 5x^2 – 2x + 24$
4. $p(x) = x^3 – 3x^2 – 4x + 12$
5. $y = x^3 – x^2 – x + 1$

Set C

1. $f(x) = x^3 – 4x^2 + x + 6$
2. $g(x) = 3x^3 – 2x^2 + 3x – 2$
3. $h(x) = 2x^3 – x^2 – 4x + 2$
4. $p(x) = 2x^4 + 9x^3 + 6x^2 – 11x – 6$
5. $y = 4x^4 – 5x^2 + 1$

Lesson 6

Determining the Zeros of Polynomial Functions by Synthetic Division

Synthetic division can also be used in determining the zeros of a polynomial function. Recall the when the remainder of a polynomial function f(x) when divided by x – c is 0, then c is a zero of f(x).

Examples:

Determine the rational zeros of the given polynomial function using synthetic division.

1. $f(x) = x^3 + 6x^2 + 11x + 6$

There are 3 zeros, real or imaginary. According to the Rational Roots Theorem, the possible rational zeros are ±1, ±2, ±3, and ±6.

If f(x) is divided by x + 1,

1	6	11	6	⌊-1
	-1	-5	-6	
1	5	6	0	

Since the remainder is 0, -1 is a zero of f(x).

If f(x) is divided by x – 1,

1	6	11	6	⌊1
	1	7	18	
1	7	18	24	

Since the remainder is not 0, -1 is not a zero of f(x).

If f(x) is divided by x + 2,

1	6	11	6	$\lfloor -2$
	-2	-8	-6	
1	4	3	0	

Since the remainder is 0, -2 is a zero of f(x).

If f(x) is divided by x – 2,

1	6	11	6	$\lfloor 2$
	2	16	54	
1	8	27	60	

Since the remainder is not 0, 2 is not a zero of f(x).

If f(x) is divided by x + 3,

1	6	11	6	$\lfloor -3$
	-3	-9	-6	
1	3	2	0	

Since the remainder is 0, -3 is a zero of f(x).

f(x) has only 3 zeros and we have already found 3. Thus the zeros are -1, -2 and -3.

2. $h(x) = 2x^4 + 3x^3 + 3x - 2$

There are 4 zeros, real or imaginary. The possible rational zeros are $\pm\frac{1}{2}$, ±1 and ±2.

If h(x) is divided by $x + \frac{1}{2}$,

2	3	0	3	-2	$\lfloor -\frac{1}{2}$
	-1	-1	1/2	-7/4	
2	2	-1	7/2	$-\frac{15}{4}$	

Since the remainder is not 0, $-\frac{1}{2}$ is not a zero of f(x).

If h(x) is divided by $x - \frac{1}{2}$,

2	3	0	3	-2	$\lfloor \frac{1}{2}$
	1	2	1	2	
2	4	2	4	0	

Since the remainder is 0, $\frac{1}{2}$ is a zero of f(x).

If h(x) is divided by x + 1,

2	3	0	3	-2	$\lfloor$-1
	-2	-1	1	-4	
2	1	-1	4	-6	

Since the remainder is not 0, -1 is not a zero of f(x).

If h(x) is divided by x – 1 ,

2	3	0	3	-2	$\lfloor$1
	2	5	5	8	
2	5	5	8	6	

Since the remainder is not 0, 1 is not a zero of f(x).

If h(x) is divided by x + 2 ,

2	3	0	3	-2	$\lfloor$-2
	-4	2	-4	2	
2	-1	2	-1	0	

Since the remainder is 0, -2 is a zero of f(x).

g(x) has 4 zeros and we have already used all possible rational zeros. We found only 2 rational zeros. This indicates that there are only 2 real rational zeros. The other 2 zeros are not real rational; they may be real irrational or imaginary. Thus the real only rational zeros are $\frac{1}{2}$ and -2.

Try this out

Determine the rational zeros of the given polynomial function using synthetic division. Leave irrational or imaginary zeros.

Set A

1. $f(x) = x^3 - 4x^2 - 2x + 5$
2. $g(x) = x^3 - 6x^2 + 11x - 6$

3. $h(x) = 2x^3 + 17x^2 + 23x – 42$
4. $p(x) = 8x^4 + 32x^3 + x + 4$
5. $y = x^4 + 2x^3 – 8x – 16$

Set B

1. $f(x) = x^3 + 6x^2 + 11x + 6$
2. $g(x) = x^3 – 7x + 6$
3. $h(x) = x^3 + x^2 – 12x$
4. $p(x) = 9x^3 – 7x + 2$
5. $y = 5x^3 + 4x^2 – 31x + 6$

Set C

1. $f(x) = x^3 – 7x^2 + 17x – 15$
2. $g(x) = 2x^3 + 3x^2 – 8x + 3$
3. $h(x) = 3x^3 + 13x^2 + 9x + 20$
4. $p(x) = x^4 + x^3 – 13x^2 – 25x – 12$
5. $y = 4x^5 + 16x^4 + 9x^3 – 9x^2$

Lesson 7

Determining the Zeros of Polynomial Functions Using Depressed Equations

Consider this division problem

$$\frac{x^3 + 6x^2 + 11x + 6}{x + 1}$$

Using synthetic division,

1	6	11	6	$\lfloor$-1	
	-1	-5	-6		
1	5	6	0	⇐	3rd line

The 3rd line indicates that x + 1 is a factor of $x^3 + 6x^2 + 11x + 6$ since the remainder is 0. Also, the 3rd line gives the quotient to the division problem which is indicated by the other entries 1, 5, and 6. These are the numerical coefficients of the quotient. That is,

$$\frac{x^3 + 6x^2 + 11x + 6}{x + 1} = x^2 + 5x + 6$$

The quotient $x^2 + 5x + 6 = 0$, when equated to 0 is called a *depressed equation* of $x^3 + 6x^2 + 11x + 6$.

Depressed equations are factors of a given polynomial. And can be used to find the roots of polynomial equation or zeros of polynomial function.

Examples:

Determine the zeros of the given polynomial function using depressed equations.

1. $f(x) = x^3 + 6x^2 + 11x + 6$

   The possible rational zeros are ±1, ±2, ±3, and ±6.

   If f(x) is divided by x + 1,

1	6	11	6	⌊-1
	-1	-5	-6	
1	5	6	0	

   Since the remainder is 0, -1 is a zero of f(x).

   The depressed equation is $x^2 + 5x + 6 = 0$

   To find the other zeros of f(x), solve the depressed equation.

   $x^2 + 5x + 6 = 0$
   $(x + 3)(x + 2) = 0$ By factoring
   $x + 3 = 0$ or $x + 2 = 0$
   $x = -3$ and $x = -2$

   Thus, the zeros of f(x) are -1, -3 and -2.

2. $g(x) = 2x^4 + 3x^3 + 3x – 2$

   The possible rational zeros are $\pm\frac{1}{2}$, ±1 and ±2.

   If h(x) is divided by x + 2,

2	3	0	3	-2	⌊-2
	-4	2	-4	2	
2	-1	2	-1	0	

   Since the remainder is 0, -2 is a zero of g(x).

The first depressed equation is $2x^3 - x^2 + 2x - 1 = 0$. This depressed equation can be used to find a second depressed equation without affecting the results.

If the depressed equation is divided by $x - \frac{1}{2}$,

2	-1	2	-1	$\lfloor \frac{1}{2}$
	1	0	1	
2	0	2	0	

Since the remainder is 0, $\frac{1}{2}$ is another zero of g(x).

The second depressed equation is $2x^2 + 2 = 0$

To find the other zeros of g(x), solve the second depressed equation.

$$2x^2 + 2 = 0$$
$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$
$$x = i \text{ or } -i$$

Recall from your lesson in quadratic equation, $\sqrt{-1}$ is an imaginary number = i.

Thus the zeros of g(x) are $\frac{1}{2}$, -2, -i and i.

Try this out

Determine the zeros of the given polynomial function using depressed equations.

Set A

1. $f(x) = x^3 - 4x^2 - 2x + 5$
2. $g(x) = x^3 - 6x^2 + 11x - 6$
3. $h(x) = 2x^3 + 17x^2 + 23x - 42$
4. $p(x) = 8x^4 + 32x^3 + x + 4$
5. $y = x^4 + 2x^3 - 8x - 16$

Set B

1. $f(x) = x^3 + 6x^2 + 11x + 6$
2. $g(x) = x^3 - 7x + 6$
3. $h(x) = x^3 + x^2 - 12x$
4. $p(x) = 9x^3 - 7x + 2$
5. $y = 5x^3 + 4x^2 - 31x + 6$

Set C

1. $f(x) = x^3 - 7x^2 + 17x - 15$

2. $g(x) = 2x^3 + 3x^2 – 8x + 3$
3. $h(x) = 3x^3 + 13x^2 + 9x + 20$
4. $p(x) = x^4 + x^3 – 13x^2 – 25x – 12$
5. $y = 4x^5 + 16x^4 + 9x^3 – 9x^2$

Lesson 8

Quadratic Surd Roots Theorem

One interesting fact about the zeros of polynomial functions or roots of polynomial equations of degree $n \geq 2$ is that there are some zeros or roots that occur in pairs. For instance, $x^2 – 3 = 0$ has roots $\sqrt{3}$ and $-\sqrt{3}$, $f(x) = x^2 – 6x + 2$ has zeros $3 + \sqrt{7}$ and $3 - \sqrt{7}$. The Quadratic Surd Roots Theorem generalizes this fact.

"If the quadratic surd $a + \sqrt{b}$ is a root of a polynomial equation, where a and b are rational numbers, and $\sqrt{b}$ is an irrational number, then $a – \sqrt{b}$ is also a root of the polynomial equation."

Examples:

If the given quadratic surd is a zero of a polynomial function, give the other quadratic surd which is also a zero of the polynomial function.

1. $2 + \sqrt{2}$

   Since $2 + \sqrt{2}$ is a zero a polynomial function, $2 - \sqrt{2}$ is also a zero of the polynomial function.

2. $4 – 3\sqrt{5}$

   Since $4 – 3\sqrt{5}$ is a zero a polynomial function, $4 + 3\sqrt{5}$ is also a zero of the polynomial function.

3. $\sqrt{3} – 7$

   Since $\sqrt{3} – 7$ is a zero a polynomial function, $-\sqrt{3} – 7$ is also a zero of the polynomial function.

4. $-9\sqrt{11} + 1$

Since $-9\sqrt{11}$ + 1 is a zero a polynomial function, $9\sqrt{11}$ + 1 is also a zero of the polynomial function.

Try this out

If the given quadratic surd is a zero of a polynomial function, give the other quadratic surd which is also a zero of the polynomial function.

Set A

1. $1 + 9\sqrt{5}$
2. $-4 - 2\sqrt{2}$
3. $\sqrt{3} - 7$
4. $3\sqrt{11} + 1$
5. $-2\sqrt{7} - 6$

Set B

1. $-5 + 7\sqrt{5}$
2. $14 - 9\sqrt{2}$
3. $2\sqrt{3} - 9$
4. $-2\sqrt{13} + 8$
5. $7\sqrt{5} - 6$

Set C

1. $8 - 2\sqrt{5}$
2. $-10 + 3\sqrt{2}$
3. $-3\sqrt{3} - 3$
4. $23\sqrt{13} - 28$
5. $-17\sqrt{5} + 16$

Lesson 9

Complex Conjugate Roots Theorem

Complex conjugate roots behave in the same manner as quadratic surd roots. That is, they also come in pairs. For instance, $x^2 + 3 = 0$ has roots $i\sqrt{3}$ and $-i\sqrt{3}$, $f(x) = 3x^2 - 4x + 5$ has zeros $\frac{2}{3} + \frac{1}{3}i\sqrt{11}$ and $\frac{2}{3} - \frac{1}{3}i\sqrt{11}$. The pairs

$i\sqrt{3}$ and $-i\sqrt{3}$ and $\frac{2}{3} + \frac{1}{3}i\sqrt{11}$ and $\frac{2}{3} - \frac{1}{3}i\sqrt{11}$ are examples of complex conjugates. The Complex Conjugate Roots Theorem generalizes the fact:

"If the complex number a + bi is a root of a polynomial equation with real coefficients, then the complex conjugate a – bi is also a root of the polynomial equation."

Examples:

If the given complex conjugate is a zero of a polynomial function, give the other complex conjugate which is also a zero of the polynomial function.

1. 2 + 3i

   Since 2 + 3i is a zero a polynomial function, 2 –3i is also a zero of the polynomial function.

2. 4 – 3i

   Since 4 – 3i is a zero a polynomial function, 4 + 3i is also a zero of the polynomial function.

3. -i – 7

   Since -i – 7 is a zero a polynomial function, -i – 7 is also a zero of the polynomial function.

4. $-9i\sqrt{11}$ + 1

   Since $-9i\sqrt{11}$ + 1 is a zero a polynomial function, $9i\sqrt{11}$ + 1 is also a zero of the polynomial function.

If the given quadratic surd is a zero of a polynomial function, give the other quadratic surd which is also a zero of the polynomial function.

Try this out

Set A

1. 1 + 9i
2. -4 – 2i
3. $i\sqrt{3}$ – 7
4. $3i\sqrt{11}$ + 1
5. $-2i\sqrt{7}$ – 6

Set B

1. $-5 + 7i$
2. $14 - 9i$
3. $2i\sqrt{3} - 2$
4. $-2i\sqrt{13} + 5$
5. $-7i\sqrt{5} - 9$

Set C

1. $9 - 2i$
2. $-12 + 3i$
3. $-4i\sqrt{3} - 8$
4. $2i\sqrt{13} - 2$
5. $-i\sqrt{5} + 6$

Lesson 10

Shortcuts in Determining the Zeros of Polynomial Functions

The different ways of determining the zeros of polynomial functions and the different theorems concerning roots of polynomial equations can be combined. This is illustrated in the next examples.

Examples:

Find all the zeros of each polynomial function.

1. $p(x) = x^3 - 2x^2 - x + 2$

Possible zeros: ±1 and ±2

1	-2	-1	2	$\lfloor$-1
	-1	3	-2	
1	-3	2	0	

Since the remainder is 0, -1 is a zero of p(x) and $x^2 - 3x + 2 = 0$ is a depressed equation. Solving the depressed equation by factoring,

$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0$$
$$x = 2 \text{ or } x = 1$$

Hence, the zeros of p(x) are -1, 2 and 1.

2. $f(x) = x^4 - 3x^3 - 4x^2 + 12x$

$$f(x) = x^4 - 3x^3 - 4x^2 + 12x$$
$$= x(x^3 - 3x^2 - 4x + 12)$$

The factored form suggests that one of the zeros is 0. The other zeros can be found from $x^3 - 3x^2 - 4x + 12 = 12$. The possible roots of this equation are ±1, ±2, ±3, ±4, ±6 and ±12.

1	-3	-4	12	⌊2
	2	-2	-12	
1	-1	-6	0	

Since the remainder is 0, 2 is a zero of f(x) and $x^2 - x - 6 = 0$ is a depressed equation. Solving the depressed equation by factoring,

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \text{ or } x = -2$$

Hence, the zeros of f(x) are 0, 2, 3 and -2.

3. $g(x) = 6x^4 + x^3 - 13x^2 - 2x + 2$

Possible zeros: ±1, ±2, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$ and $\pm\frac{2}{3}$

6	1	-13	-2	2	⌊$-\frac{1}{2}$
	-3	1	6	-2	
6	-2	-12	4	0	

Since the remainder is 0, $-\frac{1}{2}$ is a zero of g(x) and $6x^3 - 2x^2 - 12x + 4 = 0$ is the first depressed equation. Using this depressed equation to find another zero,

6	-2	-12	4	⌊$\frac{1}{3}$
	2	0	-4	
6	0	-12	0	

Since the remainder is 0, $\frac{1}{3}$ is a zero of g(x) and $6x^2 - 12 = 0$ is the second depressed equation. Solving $6x^2 - 12 = 0$,

$$6x^2 - 12 = 0$$
$$6x^2 = 12$$

$x^2 = 2$

$x = \sqrt{2}$ or $x = -\sqrt{2}$

Hence, the zeros of g(x) are $-\frac{1}{2}$, $\frac{1}{3}$, $\sqrt{2}$ and $-\sqrt{2}$.

Try this out

Find all the zeros of each polynomial function.

Set A
1. $p(x) = x^3 – 13x + 12$
2. $f(x) = x^3 + 9x^2 + 23x + 15$
3. $g(x) = 3x^3 + 9x^2 – 30x$
4. $h(x) = x^3 – 8$
5. $y = 2x^3 – 13x^2 – 26x + 16$

Set B
1. $p(x) = x^5 + x^4 – 3x^3 – x^2 + 2x$
2. $f(x) = 3x^3 – 2x^2 – 3x + 2$
3. $g(x) = 4x^3 – 13x^2 + 11x – 2$
4. $h(x) = 6x^3 + 4x^2 – 14x + 4$
5. $y = 2x^3 + 3x^2 – 8x + 3$

Set C
1. $p(x) = 6x^4 + x^3 – 13x^2 – 2x + 2$
2. $f(x) = x^5 + 3x^4 – 4x^3 – 12x^2$
3. $g(x) = 6x^4 – 19x^3 – 22x^2 + 7x + 4$
4. $h(x) = 2x^4 + 22x^3 + 46x^2$
5. $y = x^4 – 2x^3 – 15x^2 – 4x + 20$

Let's summarize

1. Every polynomial equation of a degree $n \geq 1$ has exactly n roots.

2. Zeros of polynomial functions in x are determined by equating each factor of the polynomial to 0 and then solving for x.

3. If a rational number $\frac{L}{F}$ in lowest terms is a root of the polynomial equation $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 = 0$, where $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are integers, then L is a factor of a_0 and F is a factor of a_n.

4. Any rational root of the polynomial equation $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$, where $a_{n-1}, a_{n-2}, \dots, a_2, a_1, a_0$ are integers, is an integer and is a factor of a_0.

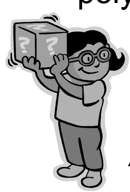
5. The Factor Theorem states that "If p(c) = 0, then x – c is a factor of p(x)." This implies that c is a zero of p(x).

6. The zeros of a polynomial function can be determined easily if the polynomial is in factored form.

7. Depressed equations are factors of a given polynomial, and can be used to find the roots of polynomial equation or zeros of polynomial function.

8. If the quadratic surd $a + \sqrt{b}$ is a root of a polynomial equation, where a and b are rational numbers, and $\sqrt{b}$ is an irrational number, then $a - \sqrt{b}$ is also a root of the polynomial equation

9. If the complex number a + bi is a root of a polynomial equation with real coefficients, then the complex conjugate a – bi is also a root of the polynomial equation.



What have you learned

1. How many zeros do the polynomial function $f(x) = x^6 - 3x^5 - x^4 + 2x^2 + x - 3$ have?
2. How many roots do the polynomial equation $2x^5 + x^4 + 8x^2 - 2x - 1 = 0$ have?
3. Determine the zeros of the polynomial function $F(x) = x(x - 2)^2(x + 3)(3x - 2)$.
4. What are the possible rational zeros of $p(x) = x^4 - 4x^3 + 2x^2 - 9$?
5. What are the possible rational roots $5x^5 - 2x^4 + x^3 - x^2 + 8x - 3 = 0$?
6. Find all the zeros of $h(x) = x^3 - 4x^2 - 7x + 10$.
7. Solve the polynomial equation $x^4 - 2x^3 - 15x^2 - 4x + 20 = 0$ using synthetic division.
8. Find all zeros of $g(x) = x^3 - 4x^2 + 5x - 2$ using depressed equations.
9. One of the roots of $x^3 - 4x^2 + 6x - 4 = 0$ is 1 + i. What are the other roots?
10. One of the zeros of $p(x) = 4x^4 + 8x^3 - 8x^2 - 4x$ is $\frac{-3+\sqrt{5}}{2}$. Find the other zeros.

Answer Key

How much do you know

1. 5
2. 4
3. 0, 3 multiplicity 2, -1 and $\frac{3}{2}$
4. ±1, ±3, ±5, ±15
5. ±1, ±$\frac{1}{3}$, ±5, ±$\frac{5}{3}$
6. 2 and 4 (multiplicity 2)
7. -2, 0, 1, and 7
8. -1, 1, and 2
9. $-2\sqrt{2}$ and 12
10. $i\sqrt{13}$, $-\frac{1}{2}$ and 1

Try this out

Lesson 1

Set A

1. 5
2. 7
3. 10
4. 6
5. 7

Set B

1. 3
2. 7
3. 8
4. 7
5. 5

Set C

1. 7
2. 6
3. 7
4. 9
5. 10

Lesson 2

Set A

1. -4, 0, 2
2. -7, 0, 1
3. $-\frac{3}{2}$, $\frac{5}{4}$, 3

4. 0, 1, $\frac{4}{3}$
5. -3, $-\frac{1}{3}$, 0

Set B

1. -2 multiplicity 3, 8 multiplicity 5
2. $-\frac{1}{5}$ multiplicity 6, $\frac{7}{2}$ multiplicity 4
3. 0, $\frac{2}{3}$ multiplicity 2, 5
4. -4 multiplicity 3, 0 multiplicity 2, $\frac{3}{2}$, $\frac{7}{3}$
5. $-\frac{2}{5}$ multiplicity 4, 0 multiplicity 3, 4, 1 multiplicity 2

Set C

1. -3, $-\frac{1}{3}$, 0, $\frac{9}{2}$
2. -1, $\frac{1}{2}$ multiplicity 2, 1
3. -5, -2, 0, $\frac{4}{3}$
4. $-\frac{5}{2}$, 0, 1, $\frac{-7}{3}$, $-\frac{1}{2}$
5. $-\frac{5}{3}$, $-\frac{3}{2}$, $-\frac{1}{4}$, 1

Lesson 3

Set A

1. ±1, ±5
2. ±1, ±2, ±3, ±6
3. ±1, ±3
4. ±1, ±2, ±4, ±8, ±16
5. ±1, $\pm\frac{1}{2}$, ±2, ±3, , $\pm\frac{3}{2}$, ±6, ±7, $\pm\frac{7}{2}$, ±14, ±21, $\pm\frac{21}{2}$, ±42

Set B

1. ±1
2. ±1, ±2, ±4, ±8, ±16, ±32
3. $\pm\frac{1}{2}$, ±1, ±3, $\pm\frac{3}{2}$
4. ±1, $\pm\frac{1}{3}$, ±2, $\pm\frac{2}{3}$, ±4, $\pm\frac{4}{3}$, ±5, $\pm\frac{5}{3}$, ±10, $\pm\frac{10}{3}$, ±20, $\pm\frac{20}{3}$

5. ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$, ±2, ±4, ±8

Set C

1. ±1, ±3, ±5, ±15
2. ±1, $\pm\frac{1}{2}$, ±2
3. ±1, $\pm\frac{1}{3}$, ±2, $\pm\frac{2}{3}$
4. ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, ±2, $\pm\frac{2}{3}$, ±4, $\pm\frac{4}{3}$, ±8, $\pm\frac{8}{3}$
5. ±1, $\pm\frac{1}{2}$, $\pm\frac{1}{3}$, $\pm\frac{1}{6}$, ±2, $\pm\frac{2}{3}$, $\pm\frac{1}{3}$

Lesson 4

Set A

1. -3, -1, 2
2. -3, -2, 1
3. -3, -2, 2
4. -2, -1, 4
5. -1, -1, 1

Set B

1. -1, 2, 3
2. -2, 3, 4
3. -2, 2, 3
4. -1, 3, 4
5. -1, 1, 5

Set C

1. -1, 1, 1
2. -2, $\frac{3}{2}$ 2,
3. –3, $\frac{2}{3}$, 3
4. -1, $-\frac{1}{2}$, $\frac{1}{2}$, 1
5. -3, -2, $-\frac{1}{2}$,1

Lesson 5

Set A

1. -2, 2, -3
2. -2, -1, -3
3. -2, -1, 4

4. -3, -2, 1
5. -1, -1, 1

Set B

1. -1, 3, 4
2. -2, 2, 3
3. -2, 3, 4
4. -2, 2, 3
5. -1, 1, 1

Set C

1. -1, 2, 3
2. $\frac{2}{3}$, i, -i
3. $\frac{1}{2}$, $\sqrt{2}$, $-\sqrt{2}$
4. -3, -2, -$\frac{1}{2}$, 1
5. -1, -$\frac{1}{2}$, $\frac{1}{2}$, 1

Lesson 6

Set A

1. 1(the only rational zero), 2 are not rational zeros
2. 1, 2, 3
3. -6, $-\frac{14}{4}$, 1
4. -4, $\frac{1}{2}$ (the 2 rational zeros), 2 are not rational zeros
5. -2, 2 (the 2 rational zeros), 2 are not rational zeros

Set B

1. -3, -2, -1
2. -3, 1, 2
3. -4, 0, 3
4. -1, $\frac{1}{3}$, $\frac{2}{3}$
5. -3, $\frac{1}{5}$, 2

Set C

1. 3 (the only rational zero), 2 are not rational zeros
2. -3, $\frac{1}{2}$, 1
3. -4 (the only rational zero), 2 are not rational zeros

4. -3, -1, -1, 4
5. -3, $-\frac{3}{2}$, 0, 0, $\frac{1}{2}$

Lesson 7
Set A

1. 1, $\frac{3+\sqrt{29}}{2}$, $\frac{3-\sqrt{29}}{2}$
2. 1, 2, 3
3. -6, $-\frac{14}{4}$, 1
4. -4, $-\frac{1}{2}$, $\frac{1+\sqrt{5}}{4}$, $\frac{1-\sqrt{5}}{4}$
5. -2, 2, $\frac{-1+i\sqrt{7}}{2}$, $\frac{-1-i\sqrt{7}}{2}$

Set B

1. -3, -2, -1
2. -3, 1, 2
3. -4, 0, 3
4. -1, $\frac{1}{3}$, $\frac{2}{3}$
5. -3, $\frac{1}{5}$, 2

Set C

1. 3, $2+\mathrm{i}$, $2-\mathrm{i}$
2. -3, $\frac{1}{2}$, 1
3. -4, $\frac{-1+i\sqrt{59}}{6}$, $\frac{-1-i\sqrt{59}}{6}$
4. -3, -1, -1, 4
5. -3, $-\frac{3}{2}$, 0, 0, $\frac{1}{2}$

Lesson 8
Set A

1. $1-9\sqrt{5}$
2. $-4+2\sqrt{2}$
3. $-\sqrt{3}-7$
4. $-3\sqrt{11}+1$
5. $2\sqrt{7}-6$

Set B

1. $-5-7\sqrt{5}$
2. $14+9\sqrt{2}$
3. $-2\sqrt{3}-9$
4. $2\sqrt{13}+8$
5. $-7\sqrt{5}-6$

Set C

1. $8+2\sqrt{5}$
2. $-10-3\sqrt{2}$
3. $3\sqrt{3}-3$
4. $-23\sqrt{13}-28$
5. $17\sqrt{5}+16$

Lesson 9
Set A

1. $1-9i$
2. $-4+2i$
3. $-i\sqrt{3}-7$
4. $-3i\sqrt{11}+1$
5. $2i\sqrt{7}-6$

Set B

1. $-5-7i$
2. $14+9i$
3. $-2i\sqrt{3}-2$
4. $2i\sqrt{13}+8$
5. $7i\sqrt{5}-9$

Set C

1. $9+2i$
2. $-12-3i$
3. $4i\sqrt{3}-8$
4. $-2i\sqrt{13}-2$
5. $i\sqrt{5}+6$

Lesson 10
Set A

1. -4, 1, 3
2. -5, -3, -1

3. -5, 0, 2
4. 2, $-1+i\sqrt{3}$, $-1-i\sqrt{3}$
5. -2, $\frac{1}{2}$, 8

Set B

1. -2, -1, 0, 1, 1
2. -1, $\frac{2}{3}$, 1
3. $\frac{1}{4}$, 1, 2
4. -2, $\frac{1}{3}$, 1
5. -3, $\frac{1}{2}$, 1

Set C

1. $-\frac{1}{2}$, $\frac{1}{3}$, $\sqrt{2}$ $-\sqrt{2}$
2. -3, -2, 0, 0, 2
3. -1, $-\frac{1}{3}$, $\frac{1}{2}$, 4
4. 0, 0, $\frac{-11+\sqrt{29}}{2}$, $\frac{-11-\sqrt{29}}{2}$
5. -2, -2, 1, 5

What have you learned

1. 6
2. 5
3. -3, 0, $\frac{2}{3}$, 2, 2
4. ±1, ±3, ±9
5. ±1, $\pm\frac{1}{5}$, $\pm\frac{3}{5}$
6. -2, 1, 5
7. -2, -2. 1, 5
8. 1, 1, 2
9. 1 – i, 2
10. 0, 1, $\frac{-3-\sqrt{5}}{2}$