Module 1
Polynomial Functions

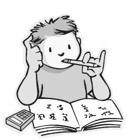
What this module is about

This module is about polynomial functions. In the previous lessons you have learned about linear and quadratic functions. These two belongs to the family of polynomials but whose degrees are 1 and 2. In this module, you will learn about functions of degree greater than 2.

What you are expected to learn

This module is designed for you to:

1. identify a polynomial function from a given set of relations,
2. determine the degree of a given polynomial function,
3. find the quotient of polynomials by,
    - algorithm
    - synthetic division
4. State and illustrate the Remainder Theorem
5. Find the value of p(x) for x = k by:
    - synthetic division
    - Remainder Theorem
6. Illustrate the Factor Theorem



How much do you know

Answer the following:

1. One of the following is not a polynomial function. Which is it?

    a. $f(x) = 4x^3 + 3x^2 + 4x – 12$
    b. $f(c) = x^3 - 6x^2 + 12x + 4$
    c. $p(x) = x^{-4} + 8x^3 – x^2 + 2x + 8$
    d. $f(x) = 7x^5 – 9x^3 + 5x –2$

2. What is the degree of the polynomial function $f(x) = 2x^4 + 3x^3 - x^2 + 5x - 4$?

3. Find the quotient and the remainder if $y = 3x^4 - x^3 + 6x^2 - 11x + 6$ divided by $3x-1$.

4. If $f(x) = x^3 + 3x^2 + 10x + 5$, what will be the value of $f(x)$ at $x = 3$?

5. What will be the value of k such that $x - 1$ is a factor of $x^3 - 3x + k$?

6. What must be the value of k such that 3 is the remainder when $f(x) = x^3 + 4x^2 - kx \div (x - 1)$?

7. What is the remainder when $f(x)= x^5 - 2x^4 + 3x^3 - 2x^2 - x + 2$ is divided by $x + 1$?

8. Which of the following binomial is a factor of $2x^3 + 5x^2 - 10x - 16$?
   a. $x - 2$
   b. $x + 2$
   c. $x - 1$
   d. $x + 1$

9. If $f(x) = 2x^4 - x^3 - 3x^2 + x - 5$, what will be the value of $f(x)$ at $x = -3$?

10. What must be the value of k so that $x + 1$ is a factor of $f(x) = 3x^3 + kx^2 - x - 2$?

What you will do

Lesson 1

Identify and Determine the Degree of the Polynomial Function from a Given Set of Relations

A function defined by $p(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$ where n is a positive integer a_n, a_{n-1}, a_{n-2} are called polynomial functions. The exponent *n* denotes the degree of the polynomial function.

The functions,

$$p(x) = 3x + 4 \text{ is of degree 1.}$$

$$p(x) = 4x^2 + 15x + 10 \text{ is of degree 2.}$$

In this lesson, you will study about polynomial functions of degree greater than 2. Remember that there are restrictions to be considered to determine if it a relation is a polynomial function. Looking back at the definition, the exponent should be positive or the value of $n > 0$.

Examples:

1. $f(x) = x^3 - 3x^2 + 4x - 12$	Polynomial of degree 3
2. $p(x) = x^4 - 4x^3 - 13x^2 + 3x + 18$	Polynomial of degree 4
3. $f(x) = x^{-3} + 4x^2 + 2x + 1$	Not a polynomial. There is a negative exponent or $n < 0$.
4. $f(x) = 4x^5 - 2x^3 + 5x - \frac{1}{x}$	Not a polynomial. There is a variable x in the denominator.

Try this out

A. Tell whether the following is a polynomial function or not.

1. $f(x) = 4x^3 + 3x^2 + 4x - 12$
2. $p(x) = x^{-4} + 8x^3 - x^2 + 2x + 8$
3. $f(x) = x^3 - 6x^2 + 12x + 4$
4. $f(x) = 7x^5 - 9x^3 + 5x - 2$
5. $p(x) = 2x^{-3} + 3x^2 + 5x - 3$
6. $p(x) = x^2 + 3x + 1 + \frac{5}{x}$
7. $f(x) = x^3 + 13$
8. $p(x) = 5x - 6 + 2\sqrt{x} + \frac{7}{x^2}$
9. $f(x) = \sqrt{2}\, x^4 + x$
10. $f(x) = 2x^4 + 3x^3 + 2x + 1$

B. Determine the degree of the polynomial function.

1. $p(x) = x^4 + 2x^3 + 2x + 1$
2. $f(x) = x^5 - x^4 + 2x^3 - 3x^2 + 4x - 12$
3. $p(x) = x^6 + 5x^5 - 6x^4 + 8x^3 + 4x^2 - 3x + 1$
4. $f(x) = x^3 + 6x^2 + 3x + 9$
5. $p(x) = x^8 + 4x^4 + 2x^2 + 1$
6. $p(x) = 16x^5 - 6$
7. $f(x) = -2x + x^2 - 5 - 2x^3$
8. $p(x) = .10x^2 + 5x^3 - 2$
9. $f(x) = 7x - 2x^4 + 1$
10. $p(x) = x^6 + 5x^3 - 6$

Lesson 2

Find the Quotient of Polynomials by Division Algorithm

Division algorithm is the division process that you are familiar with. Dividing polynomials are the same as dividing numbers.

All you have to do is to follow the steps in dividing a polynomial by another polynomial as illustrated in the example below.

Example: Divide:

1. $(-x^2 + 3x^3 - 8x + 5)$ by $(x + 2)$

$(3x^3 - x^2 - 8x + 5)$ by $(x + 2)$

$$\begin{array}{rl} & 3x^2 \\ x+2 & \overline{\big)\,3x^3 - x^2 - 8x + 5} \end{array}$$

$$\begin{array}{rl} & 3x^2 \\ x+2 & \overline{\big)\,3x^3 - x^2 - 8x + 5} \\ & \;\;3x^3 + 6x^2 \end{array}$$

$$\begin{array}{rl} & \quad 3x^2 - 7x \\ x+2 & \overline{\big)\,3x^3 - x^2 - 8x + 5} \\ & \underline{\;\;3x^3 + 6x^2 \qquad\quad} \\ & \qquad -7x^2 - 8x \end{array}$$

$$\begin{array}{rl} & \quad 3x^2 - 7x + 6 \\ x+2 & \overline{\big)\,3x^3 - x^2 - 8x + 5} \\ & \underline{\;\;3x^3 + 6x^2 \qquad\quad} \\ & \qquad -7x^2 - 8x \\ & \qquad \underline{-7x^2 - 14x} \\ & \qquad\qquad\quad 6x + 5 \\ & \qquad\qquad\quad \underline{6x + 12} \\ & \qquad\qquad\qquad -7 \end{array}$$

Steps

1. Arrange the terms of the dividend and divisor according to degree.
2. Divide the first term of the dividend by the first term of the divisor to get the first term of the quotient.
3. Multiply the result in step 2 by the divisor.
4. Subtract the result from step 3. Bring down the next term of the dividend.
5. Repeat the entire process using the result in step 4 as the new dividend.
6. Express the result as:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

$$\frac{3x^3 - x^2 - 8x + 5}{x + 2} = 3x^2 - 7x + 6 + \frac{-7}{x + 2}$$

The quotient of $-x^2 + 3x^3 - 8x + 5$ by $x + 2$ is $3x^2 - 7x + 6$ and the remainder is -7.

Check by multiplying the quotient to the divisor. Do not forget to add the remainder.

2. $(x^3 - 13x + 12)$ by $(x + 4)$

Notice the absence of an x^2 term in the dividend.

$$\begin{array}{r|l} & \underline{\;x^2\qquad\qquad\qquad\quad} \\ x+4 & x^3 + 0x^2 - 13x + 12 \end{array}$$

Add a $0x^2$ term to the dividend.

$$\begin{array}{r|l} & \underline{\;x^2\qquad\qquad\qquad\quad} \\ x+4 & x^3 + 0x^2 - 13x + 12 \\ & x^3 + 4x^2 \end{array}$$

Follow the steps in the first example.

$$\begin{array}{r|l} & \underline{\quad x^2 - 4x\qquad\qquad} \\ x+4 & x^3 + 0x^2 - 13x + 12 \\ & \underline{x^3 + 4x^2\qquad\qquad\quad} \\ & \qquad - 4x^2 - 13x \end{array}$$

Do not forget to change the sign of the subtrahend when subtracting.

$$\begin{array}{r|l} & \underline{\quad x^2 - 4x + 3\qquad} \\ x+4 & x^3 + 0x^2 - 13x + 12 \\ & \underline{x^3 + 4x^2\qquad\qquad\quad} \\ & \qquad - 4x^2 - 13x \\ & \qquad \underline{- 4x^2 - 16x\quad} \\ & \qquad\qquad\quad 3x + 12 \\ & \qquad\qquad\quad \underline{3x + 12} \\ & \qquad\qquad\qquad\; 0 \end{array}$$

$$\frac{x^3 - 13x + 12}{x + 4} = x^2 - 4x + 3$$

The quotient is $x^2 - 4x + 3$.

Try this out

Find the quotient by dividing the polynomials using division algorithm:

1. $(3x^3 - x^2 - 8x + 5) \div (x + 2)$
2. $(4x^2 + 15x + 10) \div (x - 2)$
3. $(x^3 - 2x^2 + 6x + 3) \div (x - 3)$
4. $(x^3 - 5x^2 - 9x + 3) \div (x - 4)$
5. $(2x^5 + 4x^4 + 8x - 1 \div (x + 2)$
6. $x^4 - 3x + 5) \div (x + 3)$
7. $(x^3 - 2x^2 + 4) \div (x - 3)$
8. $(-10x + 2x^4 - 5x^3 + 8) \div (x - 3)$
9. $(x^3 + 3x - 4x^2 - 12) \div (x - 4)$
10. $(x^5 + 32) \div (x + 2)$

Lesson 3

Find by Synthetic Division the Quotient and the Remainder When P(x) is Divided by (x - c)

Another method of dividing polynomials which has a very short and simple procedure is called **synthetic division**. Unlike the usual division which involves the four fundamental operations, this method requires only addition and multiplication applied to the coefficients. This method is applied when the divisor is of the form x - c.

Steps to follow in dividing by synthetic division:

1. Arrange the terms of the dividend in descending order of exponent.

2. Write the numerical coefficient in a row, with 0 representing any missing term.

3. Write the constant term c of the divisor x - c at the left hand side of the of the coefficient.

4. Bring down the leading coefficient of the dividend. Multiply it by c and add to the second column.

5. Multiply the sum obtained in step 4 by c and add to the 3rd column. Repeat this process until you reach the last column.

6. The 3rd rows of numbers are numerical coefficient of the quotient. The degree is one less than that of the dividend. The right member is the remainder.

Examples:

Find the quotients and the remainder using the steps in synthetic division. Write you answer in the form P(x) = Q(x)D(x) + R where, P(x) is the dividend, Q(x) is the quotient, Q(x) is the divisor, and R is the remainder.

1. $P(x) = x^3 + 4x^2 + 3x – 2$ by $x – 3$

$$\begin{array}{r|rrrr} & 1 & 4 & 3 & -2 \\ 3 & & 3 & 21 & 72 \\ \hline & 1 & 7 & 24 & 70 \end{array} \qquad x = 3$$

Since $Q(x) = x^2 + 7x + 24$ and $R = 70$, then

$P(x) = (x^2 + 7x + 24)(x - 3) + 70$ in the form $P(x) = Q(x)D(x) + R$

2. $P(x) = 3x^4 - 2x^3 + 5x^2 - 4x - 2$ by $3x + 1$

$\dfrac{3x^4 - 2x^3 + 5x^2 - 4x - 2}{3}$; $\quad x + \dfrac{1}{3}$

Divide both divisor and dividend by 3 then follow steps in synthetic division.

$$\begin{array}{r|rrrrr} & 1 & -\frac{2}{3} & \frac{5}{3} & -\frac{4}{3} & -\frac{2}{3} \\ -\frac{1}{3} & & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \hline & 1 & -1 & 2 & -2 & 0 \end{array} \qquad x = -\frac{1}{3}$$

$Q(x) = (x^3 - x^2 + 2x - 2)$ and $R = 0$

Therefore: $P(x) = (x^3 - x^2 + 2x - 2)(x + 1/3) + 0$

3. $P(x) = 2x^4 - 18x^2 - 7 - x^3$ by $x - 3$
$P(x) = 2x^4 - x^3 - 18x^2 + 0x - 7$

Arrange exponent in descending order and represent the missing term by 0

$$\begin{array}{r|rrrrr} & 2 & -1 & -18 & 0 & -7 \\ 3 & & 6 & 15 & -9 & -27 \\ \hline & 2 & 5 & -3 & -9 & -34 \end{array}$$

$Q(x) = 2x^3 + 5x^2 - 3x - 9$ and $R = -34$

Therefore: $P(x) = (2x^3 + 5x^2 - 3x - 9)(x - 3) - 34$

Try this out

A. Use synthetic division to divide the given polynomial P(x) by the given polynomial x – c. Write your answer in the form $P(x) = Q(x)(x-c) + R$

1. $P(x) = 4x^6 + 21x^5 - 26x^3 + 27x$ $\quad x + 5$
2. $P(x) = x^5 - 3x^4 + 4x + 5$ $\quad x - 2$
3. $P(x) = 2x^3 - 4x^2 - 5x + 3$ $\quad x + 3$
4. $P(x) = x^5 + 5x^3 - 3x + 7$ $\quad x - 2$
5. $P(x) = x^4 - 8$ $\quad x - 2$
6. $P(x) = 2x^3 + 11x + 12$ $\quad x + 4$
7. $P(x) = 2x^3 - 3x^2 + 3x - 4$ $\quad x - 2$
8. $P(x) = x^5 + 32$ $\quad x + 2$
9. $P(x) = 2x^4 - 5x^3 - 10x + 8$ $\quad x - 3$
10. $P(x) = 6x^3 - 19x^2 + x + 6$ $\quad x - 3$

B. Find the quotient and the remainder by synthetic division of the polynomial P(x) for the given polynomial x - c. Write your answer in the form P(x) = Q(x) D(x) + R

1. $P(x) = 4x^4 + 12x^3 + 9x^2 – 8x – 5$	$2x + 1$
2. $P(x) = 15x^3 – 19x^2 + 24x – 12$	$3x – 2$
3. $P(x) = -9x^4 + 9x^3 – 26x^2 + 26x – 8$	$3x - 1$
4. $P(x) = 3x^4 – x^3 + 6x^2 – 11x + 6$	$3x – 1$
5. $P(x) = 2x^3 – 5x^2 + 6x + 1$	$2x –1$
6. $P(x) = 2x^4 – x^3 + 4x^2 –12x + 3$	$2x –1$
7. $P(x) = 2x^3 – 9x^2 + 10x – 3$	$2x – 1$
8. $P(x) = 6x^3 – 2x^2 – x – 1$	$x + \frac{2}{3}$
9. $P(x) = 4x^4 – 5x^2 + 1$	$x – \frac{1}{2}$
10. $P(x) = 2x^3 + x^2 + 12$	$x + 2$

Lesson 4

State and illustrate the Remainder Theorem

In the two previous division processes illustrated, a remainder was noted when the polynomial is not exactly divisible by another polynomial. You'll get a zero remainder when a polynomial is exactly divisible by another.

By substituting the value of (c) of the divisor x – c in the polynomial P(x), you can also test whether a certain polynomial is exactly divisible by another or is a factor by the Remainder Theorem.

The Remainder Theorem states that P(c) is the remainder when the polynomial p(x) is divided by (x – c). The divisor x – c is then restated as x = c.

Examples:

1. Find the remainder using the remainder theorem if $P(x) = x^3 + 4x^2 + 3x – 2$ is divided by $x – 3$.

Solution: Instead of using synthetic division, it is easier to solve by substitution.

$P(x) = x^3 + 4x^2 + 3x – 2; x = 3$

$P(3) = (3) + 4(3)^2 + 3(3) – 2$ Substitute 3 for x.

$= 27 + 36 + 9 – 2$

$P(3) = 70$ the remainder

Hence, the polynomial $P(x) = x^3 + 4x^2 + 3x - 2$ is not exactly divisible by $x - 3$.

2. Find the value of P(x) using the remainder theorem if $P(x) = x^4 + 3x^3 - 5x^2 - 5x - 2$ is divided by $x + 2$.

Solution:

$P(x) = x^4 + 3x^3 - 5x^2 - 5x - 2$; $x = -2$
$P(-2) = (-2)^4 + 3(-2)^3 - 5(-2)^2 - 5(-2) - 2$ Substitute -2 for x.
$= 16 - 24 + 20 - 10 - 2$
$P(-2) = 0$ the remainder

Hence $P(x) = x^4 + 3x^3 - 5x^2 - 5x - 2$ is exactly divisible by $x + 2$.

You can also solve an equation using the Remainder Theorem. In the next example, the polynomial P(x) is equated to the remainder to solve for the value of k, the numerical coefficient of the x term.

Example:

Find the value of k when polynomial $3x^2 + kx + 4$ is divided by $x - 1$ and the remainder is 2.

Solution:

$3x^2 + kx + 4 = 2$	The polynomial is equal to the remainder 2.
$3(1)^2 + k(1) + 4 = 2$	Substitute 1 for x, then solve for k.
$3 + k + 4 = 2$	
$k = 2 - 7$	
$k = -5$	

Check by synthetic division

$3x^2 - 5x + 4$ Substitute k by – 5 in the original expression, $3x^2 - kx + 4$.

$$\begin{array}{r|rrr} & 3 & -5 & 4 \\ 1 & & 3 & -2 \\ \hline & 3 & -2 & 2 \end{array} \longrightarrow \text{remainder}$$

Try this out

A. Find the remainder when P(x) is divided by x – c using the remainder theorem.

1. $P(x) = (x^3 – 7x^2 + x + 10)$ $x – 2$
2. $P(x) = (x^4 + 10x^3 - 8x –80)$ $x + 10$
3. $P(x) = (x^5 + 2x^4 – 3x^3 + 4x^2 - 5x + 2)$ $x – 1$
4. $P(x) = (x^3 + 3x^2 + 10x + 5)$ $x – 3$
5. $P(x) = (x^3 + 125)$ $x + 5$
6. $P(x) = (x^3 - 4x^2 – 3x + 18)$ $x + 2$
7. $P(x) = (x^5 + 5x^3 - 3x + 7)$ $x – 2$
8. $P(x) = (x^5 + 5x^3 - 3x + 7)$ $x + 2$
9. $P(x) = (x^{59} + 3x^{35} – 5x^7 + 9x + 8)$ $x – 1$
10. $P(x) = (x^{99} - 2x^{81} + 3x^5 – 5)$ $x + 1$

B. Find the remainder when a polynomial is divided by x - c using the remainder theorem.

1. $(-x^3 + 5x^2 – 10x + 3) \div (x – 4)$
2. $(-x + 2x^3 – 3x + 3) \div (x + 2)$
3. $(-2x^3 + 3x^2 – 3x + 5) \div (x + 1)$
4. $(-9x + 2x^3 – 20) \div (x + 2)$
5. $(-3x – 15x^3 + 4x^4 + 20) \div (x –3)$
6. $(-5x^3 – 12x^2 + 10x – 6) \div (x + 3)$
7. $(-x^3 + 6x^2 – 10x + 8) \div (x – 4)$
8. $(-x^4 – 3x^3 –2x^2 + 12x + 72) \div (x + 6)$
9. $(-2x^4 – 9x^3 + 14x^2 + 68) \div (x – 2)$
10. $(-5x^5 – 3x^4 + 4x + 5) \div (x – 1)$

C. Find the value of the following function using the remainder theorem.

1. $p(x) = 2x^3 - 5x^2 + 3x -7$
   $x = -3$

2. $p(x) = 5x^3 + 7x^2 + 8$
   $x = -2$

3. $p(x) = 4x^4 + 5x^3 + 8x^2$
   $x = 4$

4. $p(x) = 3x^3 - 7x^2 + 5x –2$
   $x = -2$

5. $p(x) = 4x^3 + 2x + 10$
   $x = -3$

6. $p(x) = 5x^4 + 6x^3 + 10x^2$
   $x = 5$

7. $p(X) = 6x^2 + 3x – 9$

x = 1

8. $p(x) = 2x^3 + 4x^2 – 5x + 9$
   $x = -3$

9. $p(x) = 2x^4 - 9x^3 + 14x^2 – 8$
   $x = 2$

10. $p(x) = 2x^4 - 9x^3 + 14x^2 – 8$
    $x = -2$

D. Given a condition, determine the value of k.

1. When $kx^3 – x^2 + 2x – 30$ is divided by $(x – 2)$, the remainder is 2.
2. When $8x^3 – 4x^2 – 7x + k$ is divided by $(x – 1)$, the remainder is 5.
3. When $x^5 + x^4 – 4x^3 – 4x^2 – 8x + k$ is divided by $x – 2$, the remainder is 0.
4. When $kx^2 – x + 3$ divided by $x + 1$, the remainder is 5.
5. When $6x^2 = 4x + k$ divided by $x + 3$, the remainder is 2.

Lesson 5

Find the Value of P(x) for x = c by Synthetic Division and the Remainder Theorem

The synthetic division and remainder theorem are two ways used to find the value of P(x). You have seen in the previous lessons that the last value obtained in synthetic division is equal to the value of the remainder. Now, how is this related to the remainder theorem.

Let's find out by comparing the two processes.

Examples:

1. Use synthetic division and remainder theorem to find the value of $P(x) = x^4 – 2x^3 – x^2 – 15x + 2$ at $x =12$

Solution:

a. By synthetic division:

$$\begin{array}{r|rrrrr} & 1 & -2 & -1 & -15 & 2 \\ 12 & & 12 & 120 & 1428 & 16956 \\ \hline & 1 & 10 & 119 & 1413 & 16958 \end{array} \longleftarrow \text{ the remainder}$$

b. by remainder theorem:

$$\begin{aligned} P(12) &= x^4 – 2x^3 – x^2 – 15x + 2 \\ &= (12)^4 – 2(12)^3 – (12)^2 – 15(12) + 2 \\ &= 20{,}736 – 3456 – 144 – 180 + 2 \\ P(12) &= 16{,}958 \end{aligned}$$

Notice that the same value was obtained for the two processes. We can now say that P(x) = R, and P(x) = 16,958.

2. Use synthetic division and remainder theorem to find the value of $P(x) = 2x^3 + 8x^2 + 13x –10$ if $x = -3$

Solution:

a. by synthetic division:

$$\begin{array}{r|rrrr} & 2 & 8 & 13 & -10 \\ -3 & & -6 & -6 & -21 \\ \hline & 2 & 2 & 7 & -31 \end{array} \longleftarrow \text{the remainder}$$

P(x) = -31

b. by remainder theorem:

$$\begin{aligned} P(-3) &= 2x^3 + 8x^2 + 13x – 10 \\ &= 2(-3)^3 + 8(-3)^2 + 13(-3) – 10 \\ &= -54 + 72 - 39 -10 \\ P(-3) &= -31 \quad \text{the remainder} \end{aligned}$$

Again, notice that the value obtained using synthetic division and remainder theorem yield the same value for P(-3).

Try this out

A. Find the value of the P(x) for the given x using synthetic division and remainder theorem.

1. $P(x) = x^3 – 4x^2 + 2x – 6$ $\qquad x = 4$
2. $P(x) = x^5 – 3x^2 – 20$ $\qquad x = 2$
3. $P(x) = 2x^3 + 3x^2 – x – 79$ $\qquad x = 9$
4. $P(x) = x^3 – 8x^2 + 2x + 5$ $\qquad x = 3$
5. $F(x) = x^4 + x^3 + x^2 + x + 1$ $\qquad x = 4$

6. $P(x) = 3x^4 + 8x^2 - 1$ $x = -4$
7. $P(x) = 6x^3 + 9x^2 - 6x + 2$ $x = 2$
8. $P(x) = x^4 - 2x^3 + 4x^2 + 6x - 8$ $x = 3$
9. $P(x) = 4x^4 + 3x^3 - 2x^2 + x + 1$ $x = -1$
10. $P(x) = 2x^3 + 8x^2 - 3x - 1$ $x = -2$

B. Using synthetic division or remainder theorem, find the value of the polynomial for the given value of x.

1. $x^4 - 2x^2 - x - 6$ $x = 2$
2. $x^4 - 4x^3 + 3x^2 + 12$ $x = -3$
3. $-x^4 - x^3 + x - 5$ $x = 1$
4. $x^3 - x^2 - x - 5$ $x = 1$
5. $x^5 - 6x^3 - x - 7$ $x = -2$
6. $x^6 - x^5 - x^4 - x - 3$ $x = 2$
7. $4x^5 - 3x + 122$ $x = -2$
8. $x^5 - 4x^3 - 3x - 2$ $x = 3$
9. $x^3 - 2x^2 - 5x - 6$ $x = -1$
10. $2x^2 - 19x + 35$ $x = 7$

C. Using synthetic division or remainder theorem, find the value of y for the given x.

1. $y = 6x^3 - 17x^2 + 14x + 8$, $x = \frac{1}{3}$
2. $y = 8x^3 - 14x^2 - 5x - 1$ $x = \frac{1}{2}$
3. $y = 64x^3 + 1$ $x = -\frac{1}{4}$
4. $y = 6x^4 - 3x^2 + 1$ $x = -\frac{1}{2}$
5. $y = 4x^4 + 2x^2 + 1$ $x = \frac{1}{4}$

Lesson 6

Illustrate the Factor Theorem

In your experience with numbers, you obtain a remainder of zero when a number is exactly divisible by another number. We can say that the divisor is a factor of the dividend in that case. Same is true with polynomials.

A zero remainder obtained when applied using the Remainder Theorem will give rise to another theorem called the factor theorem. This is a test to find if a polynomial is a factor of another polynomial.

> The Factor Theorem states:
>
> Let P(x) be a polynomial. If *c* is a zero of P that is P(c) = 0, then (x - c) is a factor of P(x). Conversely, if (x - c) is a factor of P(x) then, *c* is a zero of P.

Simply, if zero is obtained as a remainder when c is substituted to the polynomial P(x), then the polynomial x – c is factor of P(x).

Examples:

1. Show that x - 2 is a factor of $x^3 + 7x^2 + 2x - 40$

Solution:

a. Using the remainder theorem

$$\begin{aligned} P(x) &= x^3 + 7x^2 + 2x - 40 \quad \text{if } x = 2 \\ &= (2)^3 + 7(2)^2 + 2(2) - 40 \\ &= 6 + 28 + 4 - 40 \\ P(x) &= 0 \end{aligned}$$

Since P(x) = 0, then x - 2 is a factor of $x^3 + 7x^2 + 2x - 40$.

b. Using another method, by synthetic division

$$\begin{array}{r|rrrr} & 1 & 7 & 2 & -40 \\ 2 & & 2 & 18 & 40 \\ \hline & 1 & 9 & 20 & 0 \end{array} \quad \longleftarrow \text{ the remainder}$$

Since the remainder is 0, then x - 2 is a factor of $x^3 + 7x^2 + 2x - 40$.

2. Determine if (x – 3) is a factor of $(2x^4 - x^3 - 18x^2 - 7)$

Solution:

a. by remainder theorem

$$\begin{aligned} P(3) &= 2x^4 - x^3 - 18x^2 - 7 \quad \text{if } x = 3 \\ &= 2(3)^4 - (3)^3 - 18(3)^2 - 7 \\ &= 2(81) - 27 - 18(9) - 7 \\ &= 162 - 27 - 162 - 7 \\ P(3) &= -34 \end{aligned}$$

Since the P(3) = -34, which is not 0 then, (x – 3) is not a factor of ($2x^4 - x^3 – 18x^2 – 7$).

b. Using synthetic division

$$\begin{array}{r|rrrrr} & 2 & -1 & -18 & 0 & -7 \\ 3 & & 6 & 15 & -9 & -27 \\ \hline & 2 & 5 & -3 & -9 & -34 \end{array} \longleftarrow \text{remainder}$$

Since r = -34, then x – 3 is not a factor of the second polynomial.

Again, we can use this knowledge to solve equations. If the polynomial x – c is a factor of P(x), then you can equate P(x) to zero. An example is given to you below.

Example:

Find the value of k so that polynomial x – 2 is the factor of $2x^3 – kx - 3$.

Solution:

By remainder theorem:

$2x^3 – kx – 3 = 0$	Since x-2 is a factor of the polynomial then equate to 0.
$2(2)^3 – k(2) – 3 = 0$	Substitute x by 2 and perform operations.
$2(8) – 2k – 3 = 0$	Solve for k.
$16 – 2k – 3 = 0$	
$-2k = - 13$	
$k = \frac{13}{2}$	

Let us check using synthetic division;

$2x^3 – 0x^2 – kx – 3$	Supply the missing term by 0
$2x^3 – 0x^2 – \frac{13}{2}x – 3$	Substitute k by $\frac{13}{2}$

$$\begin{array}{r|rrrr} & 2 & 0 & -\frac{13}{2} & -3 \\ 2 & & 4 & 8 & 3 \\ \hline & 2 & 4 & \frac{3}{2} & 0 \end{array}$$

Try this out

A. Tell whether the second polynomial is a factor of the first .

1. $P(x) = 3x^3 – 8x^2 + 3x + 2$; $(x – 2)$
2. $P(x) = 2x^4 + x^3 + 2x + 1$; $(x + 1)$
3. $P(x) = x^3 + 4x^2 + x – 6$; $(x + 3)$
4. $G(x) = 4x^3 – 6x^2 + 2x + 1$; $(2x – 1)$
5. $H(x) = x^3 – 6x^2 + 3x + 10$; $(x – 1)$

B. Answer the following:

1. Which of the following is a factor of $f(x) = x^3 – 7x + 6$
   a. $x + 2$
   b. $x – 3$
   c. $x – 1$
   d. $x + 1$

2. Which of the following is the factor of $f(x) = 2x^3 + 3x^2 – 3x – 2$
   a. $x + 2$
   b. $x – 3$
   c. $x + 1$
   d. $x – 2$

3. Which is a factor of $p(x) = 2x^4 + 3x^3 –4x^2 –3x +2$
   a. $x – 2$
   b. $x – 1$
   c. $2x + 1$
   d. $x - 3$

4. Which is a factor of $g(x) = x^3 – 2x^2 – 5x + 6$
   a. $x – 2$
   b. $x + 1$
   c. $x + 3$
   d. $x – 3$

5. Which is a factor of $p(x) = x^3 + 3x^2 – 9x – 27$
   a. $x + 3$
   b. $x + 2$
   c. $x – 2$
   d. $x – 3$

6. Which is a factor of $p(x) = 3x^3 + 2x^2 – 7x + 2$
   a. $x + 1$
   b. $x – 2$

c. $x - \frac{1}{3}$

d. $x + \frac{1}{3}$

7. Which is a factor of $p(x) = x^4 - 8x^3 + 2x^2 + 5$
   a. $x + 1$
   b. $x - 5$
   c. $x + 5$
   d. $x - 1$

8. Which is a factor of $f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$
   a. $x - 1$
   b. $x + 1$
   c. $x + 3$
   d. $x - 3$

9. Which is a factor of $f(x) = x^4 + 6x^3 + 9x^2 - 4x - 12$
   a. $x + 3$
   b. $x - 3$
   c. $x + 1$
   d. $x - 2$

10. Which is a factor of $f(x) = 2x^3 + 5x^2 + x - 2$
   a. $x + 2$
   b. $x -1$
   c. $x - 2$
   d. $x + 3$

C. Determine the value of k which is necessary to meet the given condition.

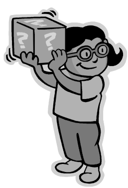
1. $(x - 2)$ is a factor of $3x^3 - x^2 - 11x + k$
2. $(x + 3)$ is a factor of $2x^5 + 5x^4 + 3x^3 + kx^2 - 14x + 3$
3. $(x + 1)$ is a factor of $-x^4 + kx^3 - x^2 + kx + 10$
4. $(x + 2)$ is a factor of $x^3 + x^2 + 5x + k$
5. $(x - 1)$ is a factor of $x^3 - x^2 - 4x + k$
6. $(x - 5)$ is a factor of $x^3 - 3x^2 - kx - 5$
7. $(x + 1)$ is a factor of $3x^3 + kx^2 - x - 2$
8. $(x + 4)$ is a factor of $kx^3 + 4x^2 - x - 4$
9. $(x + 5)$ is a factor of $kx^2 + 4x - 5$
10. $(x - 2)$ is factor of $x^3 + 3x^2 - kx + 2$

Let's summarize

1. Synthetic division is another method in finding the quotient and the remainder.

2. Remainder theorem can be used to find the value of a function, that is P(c) is the remainder when a polynomial p(x) is divided by (x- c).

3. Factor theorem: The binomial (x - a) is a factor of the polynomial P(x) if and only if P(x) = 0.



What have you learned

1. Which of the following is a polynomial function?

   a. $P(x) = 3x^{-3} – 8x^2 + 3x + 2$
   b. $P(x) = x^3 + 4x^2 + \frac{1}{x} – 6$
   c. $P(x) = 2x^4 + x^3 + 2x + 1$
   d. $G(x) = 4x^3 – \frac{6}{x^2} + 2x + 1$

2. What is the degree of the polynomial function $f(x) = 5x – 3x^4 + 1$?

3. What will be the quotient and the remainder when $y = 2x^3 – 3x^2 – 8x + 4$ is divided by (x +2)?

   a. $q(x) = 2x^2 – 7x + 6$, R = -8
   b. $q(x) = 2x^2 – 7x + 6$, R = 8
   c. $q(x) = 2x^2 – 7x – 6$, R = -8
   d. $q(x) = 2x^2 – 7x - 6$, R = 8

2. If $f(a) = 2a^3 + a^2 + 12$, what will be the value of f(a) at a = -2?
   a. 1
   b. -1
   c. 0
   d. 2

3. What must be the value of k so that when $f(x) = kx^2 - x + 3$ divide by (x + 1) and the remainder is 5?
   a. 2
   b. -2
   c. 0
   d. 1

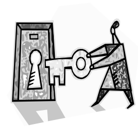
4. What must be the value of k in the function $f(x) = x^4 + x^3 – kx^2 – 25x – 12$ so that $(x – 4)$ is a factor.
   a. -12
   b. -13
   c. 13
   d. 12

5. What is the remainder when $f(x) = x^4 + 3x^2 + 4x – 1$ divided by $(x – 1)$?
   a. 7
   b. -7
   c. 6
   d. 5

6. Which of the following binomial is a factor of $f(x) = x^3 – x^2 – 5x – 3$?
   a. x + 1
   b. x + 2
   c. x -3
   d. x –2

7. If $f(x) = x^3 + 4x^2 + 3x – 2$, what will be the value f(x) at x = 3?
   a. -70
   b. 70
   c. 50
   d. –50

8. For what value of k , when $x^3 + 4x^2 – kx + 1 \div x + 1$ the remainder is 3.
   a. -1
   b. 1
   c. 2
   d. -2



Answer key

How much do you know

1. c
2. 4
3. Q(x) =(x^3 + 2x – 3), R = 3
4. 89
5. 2
6. 2
7. -5
8. a
9. 154
10. 4

Try this out

Lesson 1

A.

1. function
2. not function
3. function
4. function
5. not function
6. not function
7. function
8. not function
9. function
10. function

B.

1. fourth
2. fifth
3. sixth
4. third
5. eighth
6. fifth
7. third
8. third
9. fourth
10. sixth

Lesson 2

1. $3x^2 - 7x + 6 + -\frac{7}{x+2}$

2. $4x + 23 + \frac{56}{x-2}$

3. $x^2 + x + 9 + \frac{30}{x-3}$

4. $x^2 - x - 136 + -\frac{49}{x-4}$

5. $2x^4 + 8 + -\frac{17}{x+2}$

6. $x^3 - 3x^2 + 9x - 30 + \frac{95}{x+3}$

7. $x^2 + x + 3 + \frac{13}{x-3}$

8. $2x^3 + x^2 + 3x - 1 + \frac{5}{x-3}$

9. $x^2 + 3$

10. $x^4 - 2x^3 + 4x^2 - 8x + 16$

Lesson 3

A.

1. $4x^6 + 21x^5 - 26x^3 + 27x = (4x^5 + x^4 - 5x^3 - x^2 + 5x + 2)(x + 5) - 10$
2. $x^5 - 3x^4 + 4x + 5 = (x^4 - x^3 - 2x^2 - 4x - 4)(x - 2) - 3$
3. $2x^3 - 4x^2 - 5x + 3 = (2x^2 - 10x + 25)(x + 3) - 72$
4. $x^5 + 5x^3 - 3x + 7 = (x^4 + 2x^3 + 9x + 18x + 33)(x - 2) + 73$
5. $x^4 - 8 = (x^3 + 2x^2 + 4x + 8)(x - 2) + 8$
6. $2x^3 + 11x + 12 = (2x^2 - 8x + 43)(x + 4) - 160$
7. $2x^3 - 3x^2 + 3x - 4 = (2x^2 + x + 5)(x - 2) + 6$
8. $x^5 + 32 = (x^4 - 2x^3 + 4x^2 - 8x + 16)(x + 2) + 0$
9. $2x^4 - 5x^3 - 10x + 8 = (2x^3 + x^2 + 3x - 1)(x - 3) + 5$
10. $6x^3 - 19x^2 + x + 6 = (6x^2 - x - 2)(x - 3) + 0$

B.

1. $P(x) = (2x^3 + 5x^2 + 2x - 5) (2x + 1) + 0$
2. $P(x) = (5x^2 - 3x + 6) (3x - 2) + 0$
3. $P(x) = (-3x^3 + 2x^2 - 8x + 6) (3x - 1) - 2$
4. $P(x) = (x^3 + 2x - 3) (3x - 1) + 3$
5. $P(x) = (x^2 - 2x + 2) (2x - 1) + 3$
6. $P(x) = (x^3 + 2x - 5) (2x - 1) - 2$
7. $P(x) = (x^2 - 4x + 3) (2x - 1)$
8. $P(x) = (6x^2 - 6x + 3) (x + \frac{2}{3}) - 3$
9. $P(x) = (4x^3 + 2x^2 - 4x - 2) (x - \frac{1}{2})$
10. $P(x) = (2x^2 - 3x + 6) (x + 2) + 0$

Lesson 4

A.

1. P(2) = -8
2. P(-10) = 0
3. P(1) = 1
4. P(3) = 89
5. P(-5) 0
6. P(-2) = 0
7. P(2) = 73
8. P(-2) = -59
9. P(1) = 16
10. P(-1) = -7

B.

1. - 21
2. -5
3. 13
4. -18
5. -70
6. -9
7. 80
8. -720
9. 20
10. 1

C.

1. p(-3) = -115
2. p(-2) = -4
3. p(4) = 1,472
4. p(-2) = -64
5. p(-3) = -104
6. p(5) = 4,125
7. P(1) = 0
8. P(-3) = 6
9. P(2) = 8
10. P(-2) = 152

D.

1. k = 4
2. k = 8
3. k = 16
4. k = 1
5. k = -40

Lesson 5

A.

1. $P(x) = x^3 - 4x^2 + 2x - 6$ if $x = 4$

a. by synthetic division

$$\begin{array}{r|rrrr} & 1 & -4 & 2 & -6 \\ 4 & & 4 & 0 & 8 \\ \hline & 1 & 0 & 2 & 2 \end{array}$$

b. by remainder theorem

$x^3 - 4x^2 + 2x - 6$
$(4)^3 - 4(4)^2 + 2(4) - 6$
$64 - 64 + 8 - 6$
$= 2$

2. $P(x) = x^5 - 3x^2 - 20$ if $x = 2$
$x^5 + 0x^4 + 0x^3 - 3x^2 + 0x - 20$

a. by synthetic division

$$\begin{array}{r|rrrrrr} & 1 & 0 & 0 & -3 & 0 & -20 \\ 2 & & 2 & 4 & 8 & 10 & 20 \\ \hline & 1 & 2 & 4 & 5 & 10 & 0 \end{array}$$

a. by remainder theorem

$(2)^5 - 3(2)^2 - 20$
$32 - 12 - 20$
$= 0$

3. $P(x) = 2x^3 + 3x^2 - x - 79$ if $x = 9$

a. by synthetic division

$$\begin{array}{r|rrrr} & 2 & 3 & -1 & -79 \\ 9 & & 18 & 189 & 1692 \\ \hline & 2 & 21 & 188 & 1613 \end{array}$$

b. by remainder theorem

$2(9)^3 + 3(9)^2 - 9 - 79$
$2(729) + 3(81) - 9 - 79$
$1458 + 243 - 88$
$= 1613$

4. $P(x) = x^3 - 8x^2 + 2x + 5$ if $x = 3$

a. by synthetic division

$$\begin{array}{r|rrrr} & 1 & -8 & 2 & 5 \\ 3 & & 3 & -15 & -39 \\ \hline & 1 & -5 & -13 & -34 \end{array}$$

b. by remainder theorem

$(3)^3 - 8(3)^2 + 2(3) + 5$
$27 - 72 + 6 + 5$
$= -34$

5. $F(x) = x^4 + x^3 + x^2 + x + 1,$ if $x = 4$

a. by synthetic division

$$\begin{array}{r|rrrrr} & 1 & 1 & 1 & 1 & 1 \\ 4 & & 4 & 20 & 84 & 340 \\ \hline & 1 & 5 & 21 & 85 & 341 \end{array}$$

b. by remainder theorem

$(4)^4 + (4)^3 + (4)^2 + 4 + 1$
$256 + 64 + 16 + 5$
$= 341$

6. $P(x) = 3x^4 + 8x^2 - 1,$ if $x = -4$

a. by synthetic division

$$\begin{array}{r|rrrrr} & 3 & 0 & 8 & 0 & -1 \\ -4 & & -12 & 48 & -224 & 896 \\ \hline & 3 & -12 & 56 & -224 & 895 \end{array}$$

b. by remainder

$3(-4)^4 + 8(-4)^2 - 1$
$3(256) + 8(16) - 1$
$768 + 128 - 1$
$= 895$

7. $P(x) = 6x^3 + 9x^2 - 6x + 2$, if $x = 2$

a. by synthetic division

$$\begin{array}{r|rrrr} & 6 & 9 & -6 & 2 \\ 2 & & 12 & 42 & 72 \\ \hline & 6 & 21 & 36 & 74 \end{array}$$

b. by remainder theorem

$6(2)^3 + 9(2)^2 - 6(2) + 2$
$6(8) + 9(4) - 12 + 2$
$48 + 36 - 10$
$= 74$

8. $P(x) = x^4 - 2x^3 + 4x^2 + 6x - 8$, if $x = 3$

a. synthetic division

$$\begin{array}{r|rrrrr} & 1 & -2 & 4 & 6 & -8 \\ 3 & & 3 & 3 & 21 & 81 \\ \hline & 1 & 1 & 7 & 27 & 73 \end{array}$$

b. by remainder theorem

$(3)^4 - 2(3)^3 + 4(3)^2 + 6(3) - 8$
$81 - 54 + 36 + 18 - 8$
$27 + 54 - 8$
$= 73$

9. $P(x) = 4x^4 + 3x^3 - 2x^2 + x + 1$, if $x = -1$

a. by synthetic division

$$\begin{array}{r|rrrrr} & 4 & 3 & -2 & 1 & 1 \\ -1 & & -4 & 1 & 1 & -2 \\ \hline & 4 & -1 & -1 & 2 & -1 \end{array}$$

b. by remainder theorem

$4(-1)^4 + 3(-1)^3 - 2(-1)^2 + (-1) + 1$
$4 - 3 - 2 - 1 + 1$
$= -1$

10. $P(x) = 2x^3 + 8x^2 - 3x - 1$, if $x = -2$

a. by synthetic division

$$
\begin{array}{r|rrrr}
 & 2 & 8 & -3 & -1 \\
-2 & & -4 & -8 & 22 \\
\hline
 & 2 & 4 & -11 & 21
\end{array}
$$

b. check by remainder theorem

$2(-2)^3 + 8(-2)^2 - 3(-2) - 1$

$2(-8) + 8(4) + 6 - 1$

$-16 + 32 + 5$

$= 21$

B.

1. 0
2. 228
3. -6
4. -6
5. 11
6. 11
7. 0

8124

9. -4
10. 0

C. 1. 11

2. -6
3. 0
4. $\frac{5}{8}$
5. $\frac{69}{64}$

Lesson 6

A.

1. is a factor
2. is a factor
3. is a factor
4. is not a factor
5. is not a factor

B.

1. c
2. a

3. b
4. d
5. a
6. c
7. d
8. a
9. a
10. a

C.

1. k = 2
2. k = 13
3. k = 4
4. k = 14
5. k = 4
6. k = 9
7. k = 4
8. k = 1
9. k = 1
10. k = 11

What have you learned

1. c
2. 4
3. a
4. c
5. d
6. c
7. a
8. c
9. b
10. a