Module 4
Geometry of Shape and Size

What this module is about

This module is about polygons, specifically angles of the polygon. This module will teach you how to find the sum of the interior angles of the polygon. You will also discover other interesting facts about the interior and exterior angles of the polygon. You can also form your own generalization regarding the sum of the measures of the interior and exterior angles of a convex polygon.

What you are expected to learn

This module will help you:

1. Determine the sum of the measures of the interior and exterior angles of the triangle.
2. Determine the sum of the measures of the interior and exterior angles of a quadrilateral.
3. Make generalizations on the sum of the measures of the interior and exterior angles of a polygon.

How much do you know

1. What is the measure of each angle of an equilateral triangle?
2. If the sum of the two angles of a triangle is 130º, what is the measure of the third angle?
3. One acute angle of a right triangle is 27º. What is the measure of the other acute angle?
4. What is the measure of an exterior angle of an equilateral triangle?
5. If the measure of the vertex angle of an isosceles triangle is 50º, what is the measure of each base angles?
6. What is the sum of the measures of the interior angles of a quadrilateral?
7. How many sides does a regular polygon have if each interior angle is 120º?
8. The measures of the four angles of a pentagon are 140º, 75º, 120º, and 115º. Find the measure of the fifth angle.
9. If the radius of the circle is 12 cm, what is the diameter of the circle?
10. What is the appropriate name of the given circle?

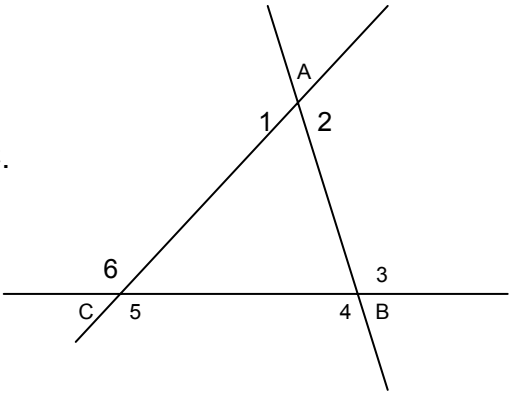
What you will do

Lesson 1

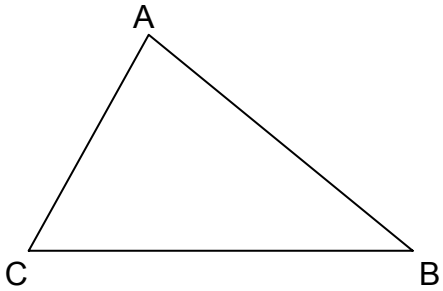
Determining the Sum of the Measures of the Interior Angles of a Triangle.

In any given triangle, say ΔABC, there are three interior angles and by observation, there are 6 exterior angles. In the given figure, $\angle A$, $\angle B$, and $\angle C$ are the interior angles of ΔABC. Each interior angle has two angles adjacent to it. For $\angle A$, $\angle 1$ and $\angle 2$, for $\angle B$, $\angle 3$ and $\angle 4$, and for $\angle C$, $\angle 5$ and $\angle 6$.

Aside from being adjacent, $\angle 1$ and $\angle 2$ are both supplementary to $\angle A$.

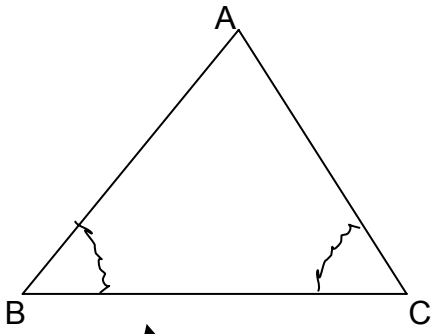


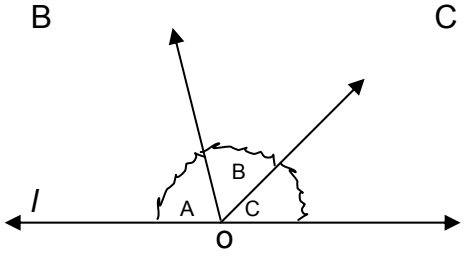
Those six angles which are adjacent to the interior angles of the triangle are called exterior angles of ΔABC. Measures of the exterior angles will be discussed in the next lesson.



To get the sum of the measures of $\angle A$ $\angle B$ and $\angle C$, you can use the protractor and then add the sum of their measures. There is another way of doing this. Here are the steps:

1. Prepare a cut out of any triangle of any size.
2. Cut off the three angles as in the figure
3. On the given line *l*, align the 3 cut out angles with all the vertices coinciding with point O.
4. All three vertices should fit perfectly on one side of the line.
5. The measure in degree of the angles about a point on the same side of a line is 180.





From the two experiment that you did, you can conclude that the $m\angle A + m\angle B + m\angle C = 180$.

Examples:

1. What is the measure of the third angle of a triangle if the measures of the two angles are
   a. 46º, 82º

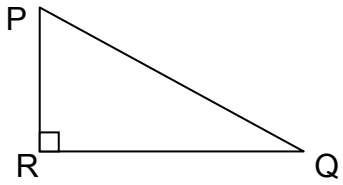
b. $51\frac{1}{2}$ º, $67\frac{1}{2}$ º

Solutions: Let x be the measure of the third angle

a. x + (46º + 82º) = 180º
   x + 128º = 180º
   x = 180º - 128º
   x = 52º, measure of the third angle

b. x + ($51\frac{1}{2}$ º + $67\frac{1}{2}$ º) = 180º
   x + 119º = 180º
   x = 180º - 119º
   x = 61º , measure of the third angle

2. The measure of one acute angle of right triangle PQR is 33º. What are the measures of the other angles?



Solution:

Since ΔPQR and ∠R = 90º , and ∠Q = 33º, then ∠P =

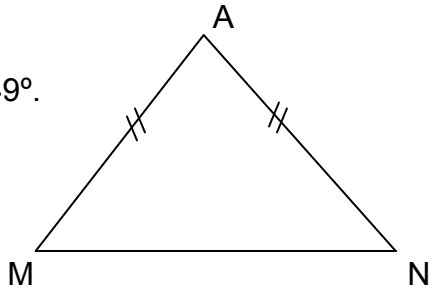
33º + 90º + ∠P = 180º
23º + ∠P = 180º
∠P = 180º - 123º
∠P = 57º

Alternate Solution:

Remember, the two acute angles of a right triangle are complementary, hence the sum of their measures is 90º. In other words, the sum of the measures of the two acute angles of a right triangle is 90. You can use this knowledge in solving the given problem.

33º + ∠P = 90º
∠P = 90º - 33º
∠P = 57º

3. In an isosceles triangle, one base angle measures 49º. Find the measures of the other two angles.



Solution: Let ΔMAN be isosceles. ∠M ≅ ∠N
m∠M = 49º
m∠N = m∠M = 49º
m∠M + m∠N + m∠A = 180º
49º + 49º + m∠A = 180º
98º + m∠A = 180º
m∠A = 180º - 98º
m∠A = 82º

Try this out

A. Given the measures of the two angles of a triangle, determine the measure of the third.

1. 36º, 70º
2. 58º, 50º
3. 90º, 25º
4. 35º, 35º
5. $64\frac{1}{2}$ º, 59º

B. Given the measure of one base angle of an isosceles triangle, determine the measures of the other two angles.

1. 45º
2. 50º
3. 53º
4. $37\frac{1}{2}$ º
5. $63\frac{1}{4}$ º

C. Given the measure of the vertex angle of an isosceles triangle, find the measure of a base angle.

1. 75º
2. 100º
3. 39º
4. 88º
5. 97º

D. In a right triangle, the measure of one acute angle is given. Find the measure of the other acute angle.

1. 45º
2. 36º
3. 52º
4. 43º
5. $61\frac{1}{2}$ º

E. Solve the following problems.

1. The measure of one acute angle of a right triangle is twice that of the other acute angle. Find the measures of the three angles.

2. Find the measures of the angles of a triangle that are in the ratio 1:2:3.

3. The vertex angle of an isosceles triangle is 15 degrees more than three times the base angle. Find the measures of all the three angles of the triangle.

4. n a triangle, one angle is ten degrees more than twice the smallest angle and the third angle is four less than three times the smallest angle. Find the measures of the three angles.

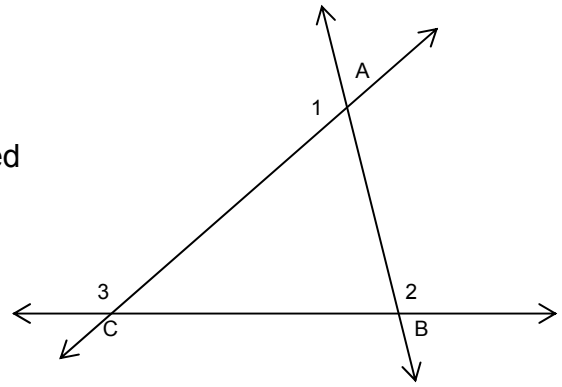
5. What are the measures of the angles of an isosceles right triangle?

Lesson 2

Sum of the Measures of Exterior Angles of a Triangle. Sum of the Interior Angles of a Quadrilateral

To find the sum of the measures of the exterior angles of a triangle, it is necessary to apply the previous lesson; the sum of the measures of the interior angles of the triangle is 180º. The phrase sum of the exterior angles of a triangle means you will use one exterior angle from each vertex of the triangle.

Thus in the given figure, to find the sum of the exterior angles of ΔABC, $\angle 1$, $\angle 2$ and $\angle 3$ are the designated exterior angles. Each of the exterior angle is adjacent to one of the interior angle of the triangle.



Study the following procedure:

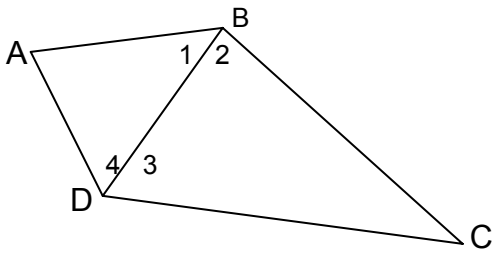
$m\angle A + m\angle B + m\angle C = 180$	Sum of the angles of a triangle is 180
$m\angle 1 + m\angle A = 180$	Definition of exterior angle of a triangle
$m\angle 2 + m\angle B = 180$	Definition of exterior angle of a triangle
$m\angle 3 + m\angle C = 180$	Definition of exterior angle of a triangle

$m\angle 1 + m\angle 2 + m\angle 3 + m\angle A + m\angle B + m\angle C = 540$	APE
$m\angle A + m\angle B + m\angle C = 180$	by subtraction
$m\angle 1 + m\angle 2 + m\angle 3 = 360$	

Therefore, the sum of the measures of the exterior angles of a triangle one on each vertex is equal to 360.

What about a quadrilateral? How do you get the sum of the measures of the interior angles of a four-sided polygon?

Consider quadrilateral ABCD. What is the sum of the measures of $\angle A$, $\angle B$, $\angle C$ and $\angle D$? To determine this, draw a diagonal, say BD. Two triangles are formed, ΔABD and ΔBCD.



In each triangle, the sum of the interior angles is 180.

To determine the sum of the four angles, study the following procedures.

In ΔABD, $m\angle A + m\angle 1 + m\angle 4 = 180$
In ΔBCD, $m\angle C + m\angle 2 + m\angle 3 = 180$
$m\angle A + m\angle 1 + m\angle 2 + m\angle C + m\angle 3 + m\angle 4 = 360$ by APE

But $m\angle B = m\angle 1 + m\angle 2$ and
$m\angle D = m\angle 3 + m\angle 4$

By substitution, $m\angle A + m\angle B + m\angle C + m\angle D = 360$.

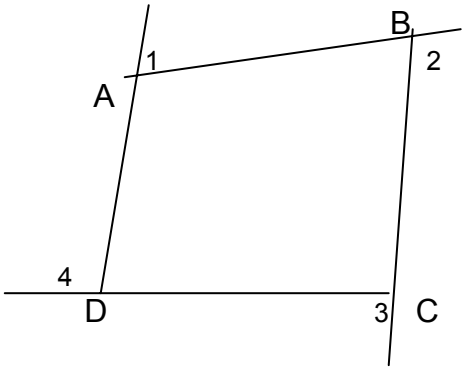
Therefore, the conclusion is that, the sum of the measures of the interior angles of a quadrilateral is 360.

How about the sum of the exterior angles of a quadrilateral? Is it the same as that of the sum of the exterior angles of the triangle?

Exploration:

Given the figure at the right. Quadrilateral ABCD has one exterior angle in each vertex. Those are $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$. Let us name the interior angles, $\angle A$, $\angle B$, $\angle C$ and $\angle D$.

To determine the sum of the measures of $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$, study the following steps.



From the previous paragraph, it is known that
$m\angle A + m\angle B + m\angle C + m\angle D = 360$.

By definition of exterior angle,

$m\angle A + m\angle 1 = 180$
$m\angle B + m\angle 2 = 180$
$m\angle C + m\angle 3 = 180$
$m\angle D + m\angle 4 = 180$

By Addition Property of Equality or APE

$(m\angle A + m\angle B + m\angle C + m\angle D) + (m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4) = 720$
Subtract: $(m\angle A + m\angle B + m\angle C + m\angle D) = 360$
$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 360$

Therefore, the sum of exterior angles of a quadrilateral is equal to 360.

Examples:

1. The measures of the three angles of a quadrilateral are 75º, 101º and 83º. Find the measure of the fourth angle.

Solution:

Let x = the measure of the fourth angle.

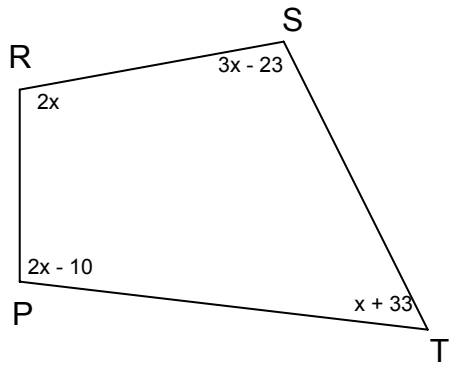
$x + 75^\circ + 101^\circ + 83^\circ = 360^\circ$

$x + 259^\circ = 360^\circ$

$x = 360^\circ - 259^\circ$

$x = 101^\circ$

2. Given quadrilateral PRST. Using the figure, find x, $m\angle P$, $m\angle R$, $m\angle S$ and $m\angle T$.



Solution: Write the representations for each angle.

$m\angle P = 2x - 10$

$m\angle R = 2x$

$m\angle S = 3x - 23$

$m\angle T = x + 33$

Using the representations for each angle, substitute to the equation

$m\angle P + m\angle R + m\angle S + m\angle T = 360$

$(2x - 10) + (2x) + (3x - 23) + (x + 33) = 360$

$8x + 33 - 33 = 360$

$8x + 0 = 360$

$8x = 360$

$x = \frac{360}{8}$

$x = 45$

$m\angle P = 2x - 10$

$= 2(45) - 10$

$= 90 - 10$

$m\angle P = 80$

$m\angle R = 2x$

$= 2(45)$

$m\angle R = 90$

$m\angle S = 3x - 23$

$= 3(45) - 23$

$= 135 - 23$

$m\angle S = 112$

$m\angle T = x + 33$

$= 45 + 33$

$m\angle T = 78$

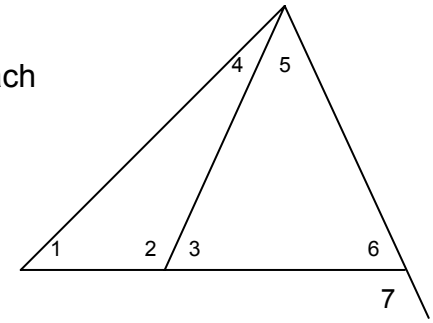
To check:

$m\angle P + m\angle R + m\angle S + m\angle T =$ _____

$80 + 90 + 112 + 78 = 360$

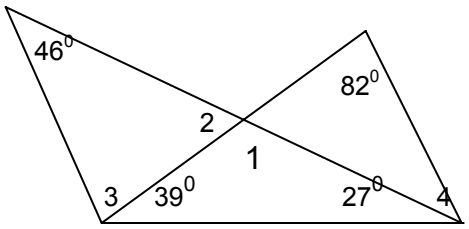
Try this out

A. Use the given information to find the measure of each numbered angle in the figure. $\angle 1 \cong \angle 5$.



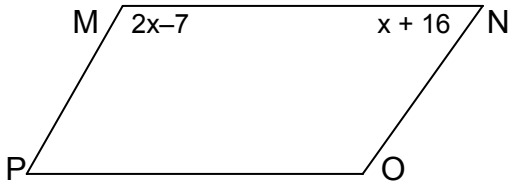
1. $m\angle 2 = 106$, $m\angle 6 = 85$
2. $m\angle 1 = 43$, $m\angle 7 = 112$
3. $m\angle 2 = 115$, $m\angle 4 = 31$
4. $m\angle 5 = 51$, $m\angle 3 = 72$

B. In the figure, determine the measure of the numbered angles.

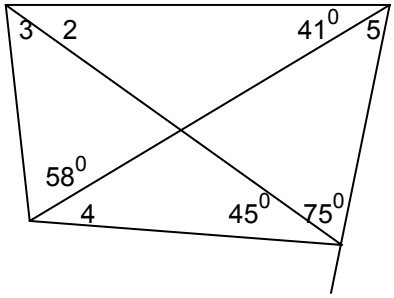


4. $m\angle 1$
5. $m\angle 2$
6. $m\angle 3$
7. $m\angle 4$

C. In quadrilateral MNOP, $\angle M \cong \angle O$, $\angle N \cong \angle P$. Using the given in the figure, find the measures of all the angles.



D. Use the given information to find the measures of the following angles.



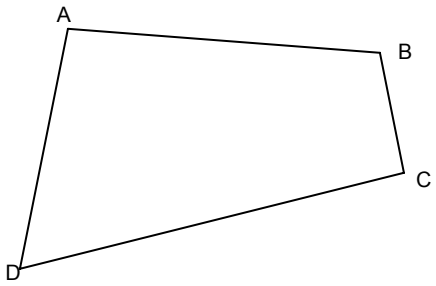
1. $\angle 1$
2. $\angle 2$
3. $\angle 3$
4. $\angle 4$
5. $\angle 5$
6. $\angle 6$

Lesson 3

Finding the Sum of the Interior Angles and Exterior Angles of Any Polygon.

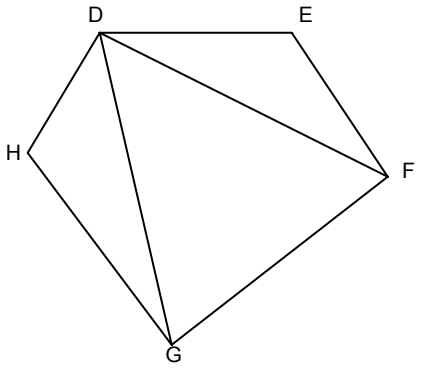
To determine the sum of the measures of interior angles and exterior angles of any polygon (polygon of *n* sides) like the given pentagon, you have to find out the number of triangles that can be formed by drawing the diagonals without intersecting each other except at their endpoints.

In quad ABCD, draw diagonal AC.



With the introduction of diagonal AC, two triangles are formed. The same thing happens if diagonal BD is drawn. If the sum of the measures of the interior angles of each triangle is 180, then for quadrilaterals, the sum of the interior angles is 360.

The polygon at the right has five (5) sides. If the diagonal is drawn from one vertex, say D, only two diagonals can be drawn, $\overline{DF}$ and $\overline{DG}$.



With these two diagonals, three non-overlapping triangles are formed, ΔDEF, ΔDFG and ΔDGH.

If in each triangle, the sum of the measures of interior angles is 180, then in pentagon DEFGH, the sum of the measures of the five angles is equal to 3(180) or 540.

This findings can be summarized using the formula $S_a = (n - 2)\ 180$, where S_q is the sum of the measures of interior angles of the polygon, *n* is the number of sides of the polygon. This formula can be applied to convex polygons only.

You also have to remember this: The sum of the measures of interior angles of a regular polygon with *n* sides is equal to $S_a = (n - 2)\ 180$.

Regular polygons are polygons which are both equilateral and equiangular.

A polygon is *convex* if and only if the lines containing the sides of the polygon do not contain points in its interior.

To get the measure of each interior angle of a regular polygon you simply remember this. The measure of each interior angle of a regular polygon with *n* sides is equal to $I_a = \frac{(n-2)180}{n}$, where, I_a is an interior angle, and *n* is the number of sides of the polygon.

If you can get the measure of each interior angle of a regular polygon, then you can also compute for the measure of each exterior angle of a regular polygon. You only have to remember these formulas.

From the generalizations that you had made in the earlier part of the lesson, the sum of the measures of the exterior angle of the convex polygon is equal to 360. This includes the regular polygons too.

The measure of each exterior angle of a regular polygon with n sides is given by the formula: $E_a = \frac{360}{n}$, where E_a is an exterior angle of a regular polygon, and *n* is the number of sides of the polygon.

Examples:

1. What is the sum of the measures of the interior angles of a convex polygon with

   a. 11 sides
   b. 15 sides

Solutions:

a. $S_a = (n - 2)\ 180$
$= (11 - 2)\ 180$
$= 9(180)$
$= 1620$

b. $S_a = (n - 2)\ 180$
$= (15 - 2)\ 180$
$= 13(180)$
$= 2340$

2. How many sides does a convex polygon have if the sum of the measures of its interior angles is 1440?

Solution:

$S_a = (n - 2)\ 180$
$1440 = (n - 2)\ 180$
$1440 = 180n - 360$
$n = \frac{1440 + 360}{180}$
$n = \frac{1800}{180}$
$n = 10$

The polygon has 10 sides.

3. Find the sum of the measures of the interior angles of a convex heptagon.

Solution:

$S_a = (n - 2)\ 180$
$= (7 - 2)\ 180$
$= 5(180)$
$= 900$

4. Find the measure of each interior angle of a regular 11-sided polygon.

Solution:

Measure of each interior angle is $I_a = \frac{(n-2)180}{n}$

$$I_a = \frac{(11-2)(180)}{11}$$

$$I_a = \frac{9(180)}{11}$$

$$I_a = \frac{1620}{11}$$

5. How many degrees are there in each of the exterior angle of a regular hexagon.

Solution:

$$E_a = \frac{360}{n}$$

$$E_a = \frac{360}{6}$$

$$= 60$$

6. If each exterior angle of a polygon is 36º, how many sides does the polygon have?

Solution:

$$E_a = \frac{360}{n}$$

$$36 = \frac{360}{n}$$

$$36n = 360$$

$$n = \frac{360}{36}$$

$$n = 10$$

7. If each interior angle of a polygon is 150º, how many sides does the polygon have?

Solution:

$$m + 150 = 180$$

$$m = 180 - 150$$

$$m = 30$$

Using the figure, with m as the exterior angle of the polygon,

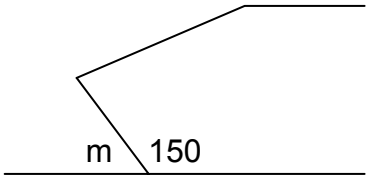
$$E_a = \frac{360}{n}$$

$30 = \frac{360}{n}$

$30n = 360$

$n = \frac{360}{30}$

$n = 12$



Try this out

A. Complete the table below.

	Polygon	No. of Sides	No. of Vertices	No. of Diagonals from one vertex	No of triangles	Sum of the Measures of interior Angles
1	Quadrilateral					
2	Pentagon					
3	Hexagon					
4	Nonagon					
5	Dodecagon					
6	N-gon					

B. Find the number of sides in each regular polygon with exterior angle of the given measure.

1. 90º
2. 45º
3. 36º
4. 60º

C. How many sides does a regular polygon have if each angle measures:

5. 120º
6. 90º
7. 135º
8. 150º

D. Find the sum of the measures of the exterior angles one at each vertex, of each of the following convex polygon.

9. decagon
10. hexagon
11. octagon
12. dodecagon

E. The sum of the measures of the interior angles of a polygon is given. Find the number of sides of the convex polygon.

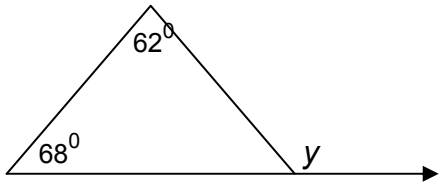
13. 1260º
14. 1620º
15. 2520º
16. 3060º

F. The sum of the measures of the interior angles of a regular polygon is given. Find the measure of each angle.

17. 1260º
18. 2520º
19. 3060º
20. 720º

G. Solve the following problems.

21. Find y in the given figure.



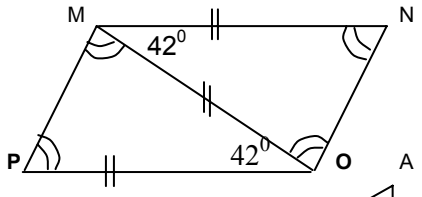
22. The measures of the two angles of a quadrilateral are 105º and 107º. If the remaining two angles are congruent, find the measure of each angle.

23. In a convex quadrilateral WXYZ, the measure of $\angle X$ is twice the measure of $\angle Y$. If $\angle Y \cong \angle Z \cong \angle W$, find the measure of each angle.

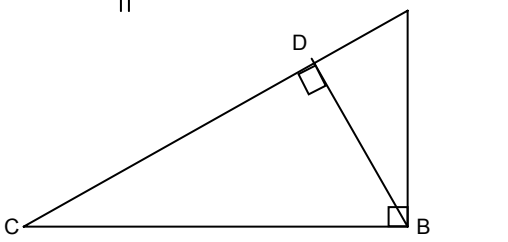
24. The number of diagonals from a vertex in a regular polygon is 7. How many sides does the polygon have? What is the measure of each interior angle?

25. Each interior angle of a regular polygon is twice the measure of each exterior angle. How many sides does the polygon have?

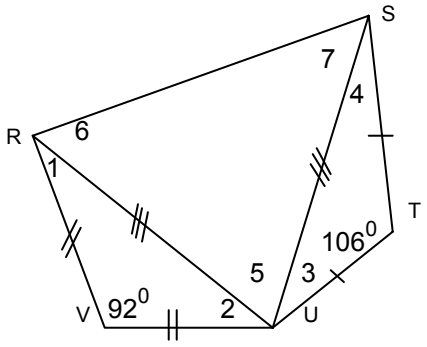
26. In the figure, like markings indicate congruent parts. Find the measure of each unknown angle.



27. $\angle ABC$ is a right angle, $\angle CDB$ is a right angle. If the $m\angle C = 32$, find the measures of the following angles:
    $\angle A$
    $\angle ADB$
    $\angle ABD$
    $\angle CBD$



28. In the given figure, RV = UV, RU = SU, ST = TU, $m\angle RVU = 92$, $m\angle STU = 106$, $m\angle U = 137$, find the measures of all the numbered angles.



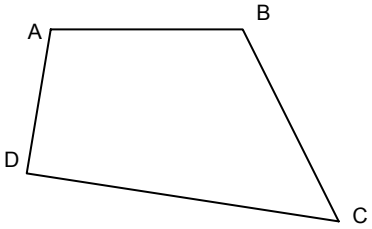
Let's summarize

1. The sum of the measures of the interior angles of a triangle is 180.
2. The sum of the measures of the interior angles of a quadrilateral is 360.
3. The sum of the measures of the interior angles of a convex polygon is given by the formula (n -2)180, where n is the number of sides of the polygon.
4. The sum of the measures of the exterior angle one on each vertex of a polygon is 360.
5. The measure of each interior angle of a regular polygon of n sides is given by the formula $I_a = \frac{(n-2)180}{n}$, where n is the number of sides of the polygon.
6. The measure of each exterior angle of a regular polygon of n sides is given by the formula $E_a = \frac{360}{n}$, where n is the number of sides of the polygon.

What have you learned

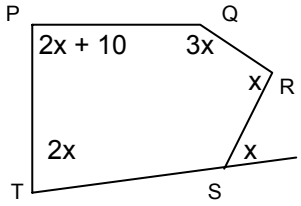
Answer each question as indicated.

1. What is the measure of each angle of a regular pentagon?
2. The sum of the measures of the two angles of a triangle is 127. What is the measure of the third angle?
3. The measure of one base angle of an isosceles triangle is 67. Find the measure of the vertex angle.
4. Given quadrilateral ABCD. If m$\angle$C = 60, and if $\angle A \cong \angle B$, what is the m$\angle$D if its measure is 27 less than the measure of $\angle$A.



5. What is the sum of the measures of the interior angles of an octagon?
6. If each interior angle of a polygon is 160º, how many sides does the polygon have?

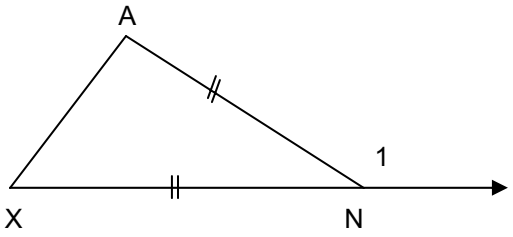
7. Find the value of x using the given in the figure



8. The angles of a triangle are in the ratio 1:3:5. Find the measures of the three angles.

9. If each interior angle of a regular polygon measures 150º, how many sides does the polygon have?

10. In the figure, $\overline{AN} \cong \overline{XN}$. If $\angle$A = 62º, find the measure of $\angle$1.



Answer Key

How much do you know.

1. 60º
2. 50º
3. 63º
4. 120º
5. 65º
6. 360º
7. 6
8. 90
9. 24 cm
10. circle A

Try this out

Lesson 1

A.
1. 74º
2. 72º
3. 65º
4. 110º
5. 56 $\frac{1}{2}$ º

B.
1. 45º, 90º
2. 50º, 80º
3. 53º, 74º
4. 37 $\frac{1}{2}$ º, 105º
5. 63 $\frac{1}{4}$ º, 53 $\frac{1}{2}$ º

C.
1. 52 $\frac{1}{2}$ º
2. 40º
3. 70 $\frac{1}{2}$ º
4. 46º
5. 41 $\frac{1}{2}$ º

D. 1. 45º
2. 54º
3. 38º
4. 47º
5. $28\frac{1}{2}$ º

E. 1. 30º, 60º, 90º
2. 30º, 60º, 90º
3. 33º, 33º, 114º
4. 29º, 68º, 83º
5. 45º, 45º, 90º

Lesson 2

A. 1. $m\angle 3 = 74$, $m\angle 4 = 53$, $m\angle 5 = 21$, $m\angle 1 = 21$, $m\angle 7 = 95$

2. $m\angle 2 = 11$, $m\angle 3 = 69$, $m\angle 4 = 26$, $m\angle 6 = 68$, $m\angle 5 = 43$

3. $m\angle 1 = 34$, $m\angle 3 = 65$, $m\angle 5 = 34$, $m\angle 6 = 81$, $m\angle 7 = 99$

4. $m\angle 1 = 51$, $m\angle 2 = 108$, $m\angle 4 = 21$, $m\angle 6 = 57$, $m\angle 7 = 123$

B. 5. $m\angle 1 = 114$
6. $m\angle 2 = 66$
7. $m\angle 3 = 68$
8. $m\angle 4 = 32$

C. $m\angle M = m\angle O = 2(57) - 7$
$= 107$

$m\angle P = m\angle N = 57 + 16$
$= 73$

D. 1. $m\angle 1 = 120$
2. $m\angle 2 = 19$
3. $m\angle 3 = 62$
4. $m\angle 4 = 15$
5. $m\angle 5 = 45$
6. $m\angle 6 = 60$

Lesson 3

A.

	Polygon	No. of Sides	No. of Vertices	No. of Diagonals from one vertex	No of triangles	Sum of the Measures Of interior Angles
1	Quadrilateral	4	4	1	2	360
2	Pentagon	5	5	2	3	540
3	Hexagon	6	6	3	4	720
4	Nonagon	9	9	6	7	1260
5	Dodecagon	12	12	9	10	1800
6	N - gon	n	N	N – 3	N – 2	(n – 2)180

B. 1. 4 sides
2. 8 sides
3. 10 sides
4. 6 sides

C. 5. 6 sides
6. 4 sides
7. 8 sides
8. 12 sides

D. 9. 360
10. 360
11. 360
12. 360

E. 13. 9 sides
14. 11 sides
15. 16 sides
16. 19 sides

F. 17. 140º
18. 157.5º
19. 161.05º
20. 120º

G. 21. y = 130º
22. Each of the remaining angle measures 74º.
23. m∠W = 72, m∠X = 144, m∠Y = 72, m∠Z = 72
24. 10 sides, 144
25. 6 sides
26. m∠N = 69, m∠MON = 69, m∠P = 69, m∠PMO = 69

27. $m\angle A = 58$, $m\angle ADB = 90$, $m\angle ABD = 32$, $m\angle CBD = 58$
28. $m\angle 1 = 44$, $m\angle 2 = 44$, $m\angle 3 = 37$, $m\angle 4 = 37$, $m\angle 5 = 56$, $m\angle 6 = 62$
    $m\angle 7 = 62$

What have you learned

1. 108º
2. 53º
3. 46º
4. 82º
5. 1080º
6. 18 sides
7. x = 50º
8. 20º, 60º, 100º
9. 12 sides
10. $m\angle 1 = 124$