Module 3
Plane Coordinate Geometry

What this module is about

This module will show you a different kind of proving. The properties of triangles and quadrilaterals will be verified in this module using coordinate plane and the application of the lessons previously discussed in the other modules.

Furthermore, you will also have the chance to do analytical proof and compare it with the geometric proof that you have been doing since the start of the year. Included in this module are lessons that will discuss in detail properties/relationships of circles with other figures in the coordinate plane.

What you are expected to learn

This module is written for you to

1. Define and illustrate coordinate proof.
2. Verify some properties of triangles and quadrilaterals by using coordinate proof.
3. Determine the difference between geometric proofs and coordinate proof.
4. Illustrate the general form of the equation of the circle in a coordinate plane.
5. Derive the standard form of the equation of the circle from the given general form.
6. Find the coordinate of the center of the circle and its radius given the equation .
7. Determine the equation of a circle given its

    a. center and radius
    b. radius and the point of tangency with the given line

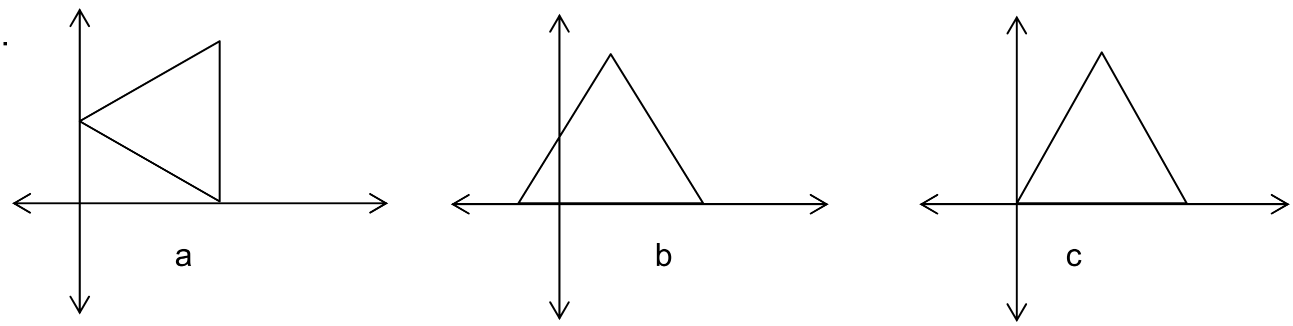
8. Analyze and solve problems involving circles.

How much do you know

Answer the following questions

Tell which of the axes placements will simplify a coordinate proof for a theorem involving the figure shown.

1.



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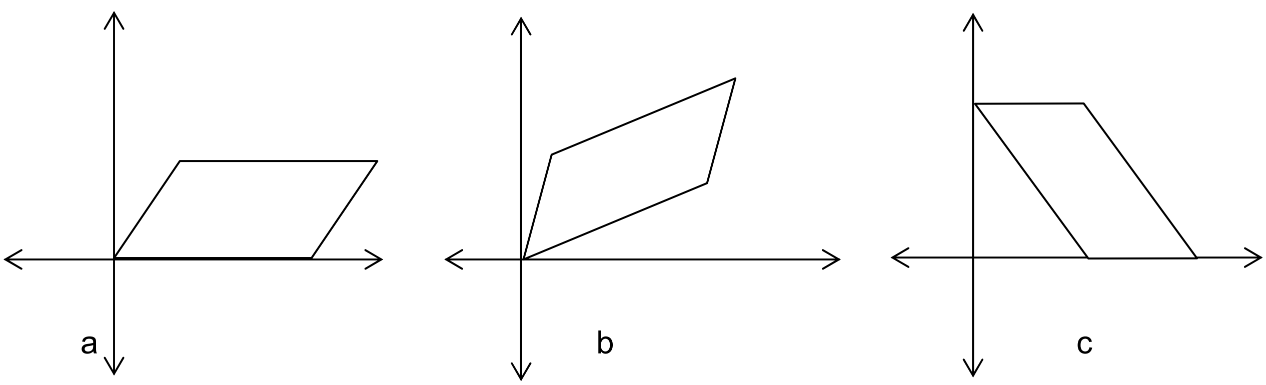

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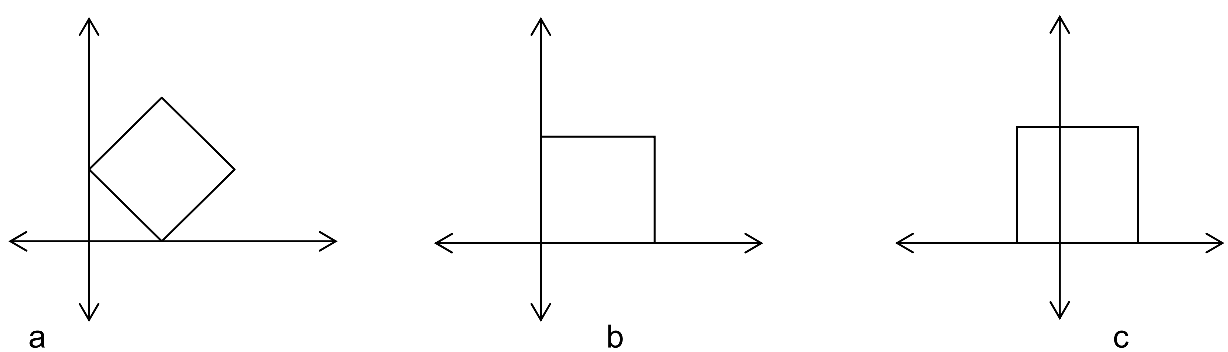


Give the center and radius of the following circles:

4. $x^2 + y^2 = 16$
5. $x^2 + y^2 – 25 = 0$
6. $(x – 3)^2 + (y + 1)^2 = 36$
7. $x^2 + y^2 – 4x + 10y + 16 = 0$
8. Give the standard form of the circle whose center is at (2, -1) and a radius of 7 units.

Write the equation of the circle satisfying the following conditions.

9. The line segment joining (4, -2) and (-8, -6) is a diameter.
10. The center is at (0, 5) and the circle passes through (6, 1).

What you will do

Lesson 1

Coordinate Proof

The coordinates of a point on the coordinate plane are real numbers, thus it is possible to prove theorems on geometric figures by analytic or algebraic method. We call this proof the coordinate proof.

In writing coordinate proof, some suggestions have to be taken into considerations. First, we may choose the position of the figure in relation to the coordinate axes. If the figure is a polygon, it is always simpler to put one of the sides on either the x-axis or the y-axis.

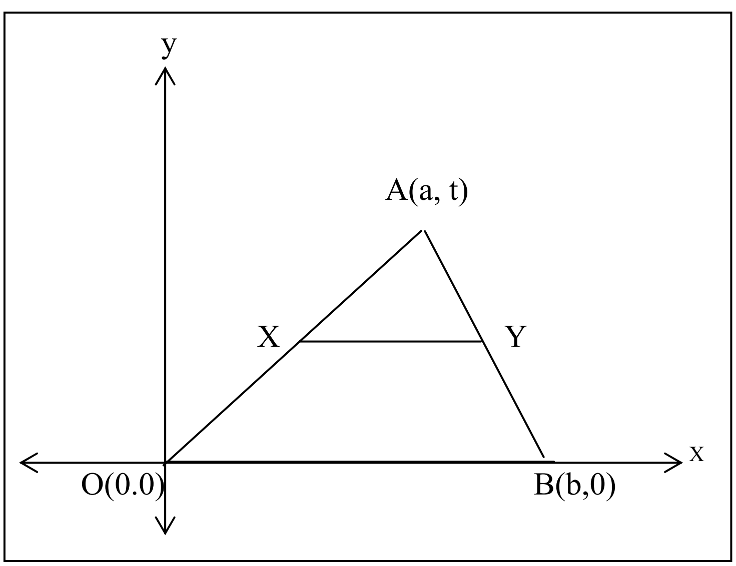
Second, we may place one of the vertices on the origin.

Third, the essential properties of the given figure should be expressed by the coordinates of key points. The proof is accomplished by setting up and simplifying algebraically equations and relations involving these coordinates.

Fourth, the figure should never be made special in any way so that the proof will be general and can be applied to all cases. Except for zero (0), numerical coordinate should not be used in the proof.

Example 1. Prove analytically that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and measures one-half of it.

Solution:



a. Analysis: The figure shows the triangle appropriately placed on rectangular coordinate plane. We have to show two properties here.

   1) $\overline{XY} \parallel \overline{OB}$ and (2) XY = $\frac{1}{2}$ OB.

b. Since X and Y are midpoints, then the midpoint formula can be used to get their coordinates. The coordinates of X are $\left(\frac{a}{2},\frac{t}{2}\right)$ and that of Y are $\left(\frac{a+b}{2},\frac{t}{2}\right)$.

   The slope m_1 of OB = $\frac{0-0}{b-0}=0$. The slope m_2 of XY is

$$m_2 = \frac{\frac{t}{2}-\frac{t}{2}}{\frac{a}{2}-\frac{a+b}{2}} = \frac{0}{\frac{-b}{2}}=0$$

Since the slope of OB = 0 and that of XY = 0, then the two slopes are equal. Therefore, $\overline{XY} \parallel \overline{OB}$. Therefore, the segment joining the midpoints of the two sides of a triangle is parallel to the third side.

c. For the second conclusion, get the length of XY and OB.

$$XY = \sqrt{\left(\frac{a+b}{2}-\frac{a}{2}\right)^2+\left(\frac{t}{2}-\frac{t}{2}\right)^2}$$

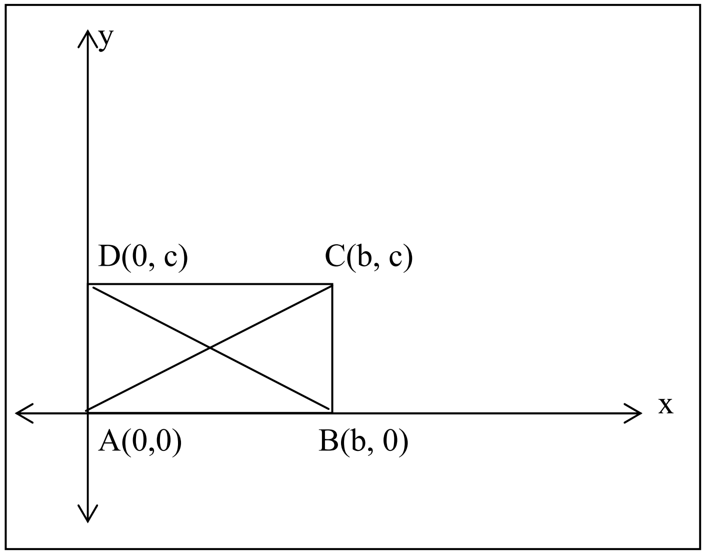
$$= \sqrt{\left(\frac{b}{2}\right)^2}$$

$$= \frac{b}{2}$$

$$OB = \sqrt{(b-0)^2+(0-0)^2}$$
$$= \sqrt{b^2}$$
$$= b$$

The computation showed that XY = $\frac{b}{2}$ and OB = *b.* Therefore, XY = $\frac{1}{2}$OB. Hence, the segment joining the midpoints of the two sides of a triangle is one-half the third side.

Example 2. Prove that the diagonals of a rectangle are equal.



Given: ABCD is a rectangle
Prove: AC = BD

Proof:

In this problem we are given not just any quadrilateral but a rectangle. Place one vertex on the origin. In the figure, it was vertex A which was in the origin. Then vertex B on the x-axis, and vertex D on the y-axis.

The following are the coordinates of the vertex : A(0, 0), B(b, 0), C(b, c) and D(0, a). AC and BD are the two diagonals. We have to determine if the length of AC is equal to the length of BD.

$$AC = \sqrt{(b-0)^2 + (c-0)^2}$$

$$= \sqrt{b^2 + c^2}$$

$$BD = \sqrt{(0-b)^2 + (c-0)^2}$$

$$= \sqrt{(-b)^2 + c^2}$$

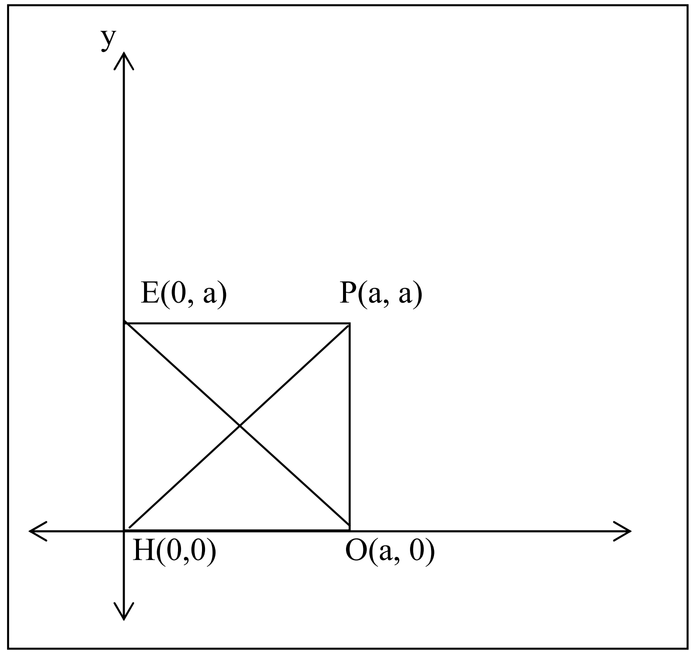
$$= \sqrt{b + c^2}$$

Therefore, from the computed distances or lengths, AC = BD. Hence, the diagonals of rectangle are equal.

Example 3. Prove that the diagonals of a square are perpendicular to each other.

Given: HOPE is a square

Prove: HP $\perp$ OE



Proof:

To show the perpendicularity of $\overline{HP}$ and $\overline{OE}$, then their slopes must be the negative reciprocal of each other. Since the vertices of the square were placed on the coordinate plane in such a way that makes proving simpler, then we are now ready to find the slopes of both $\overline{HP}$ and $\overline{OE}$.

Computing for the two slopes ,

$$m\ \overline{HP} = \frac{a-0}{a-0} = \frac{a}{a} = 1$$

$$m\ \overline{OE} = \frac{0-a}{a-0} = \frac{-a}{a} = -1$$

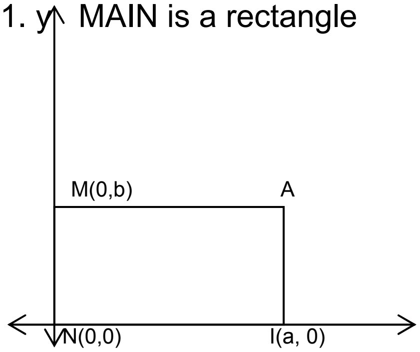
From the results you can easily see that the reciprocals of $\overline{HP}$ and $\overline{OE}$ are 1 and – 1 which are negative reciprocals.

Therefore, $\overline{HP}$ and $\overline{OE}$ are perpendicular to each other.

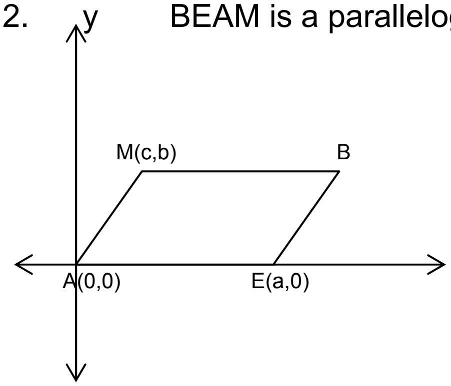
Try this out

A. Determine the coordinates of A, B or C.

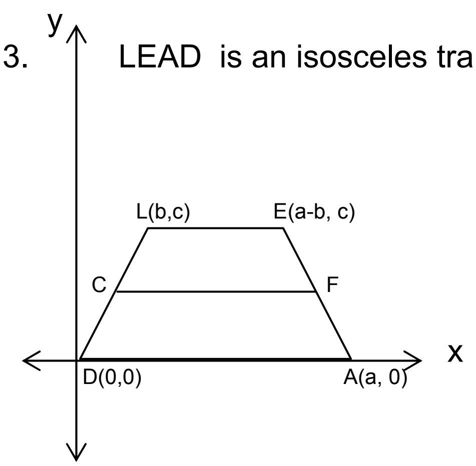
1. MAIN is a rectangle



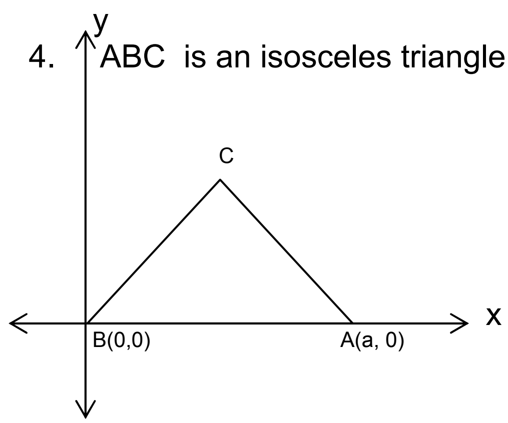
2. BEAM is a parallelogram



3. LEAD is an isosceles trapezoid



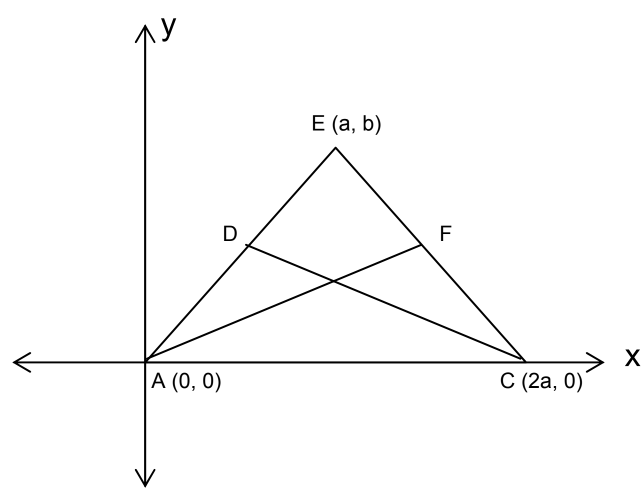
4. ABC is an isosceles triangle



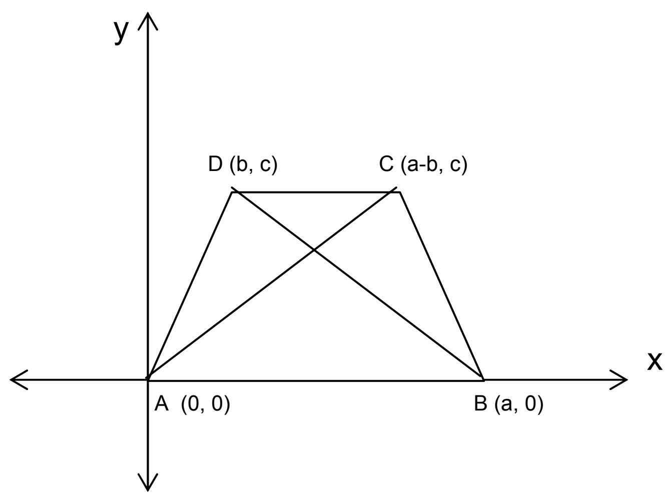
B. Given the following statements, do the following:

a. Use the given figure on the rectangular coordinate plane.
b. Write the hypothesis.
c. Write what is to be proven.
d. Prove analytically.

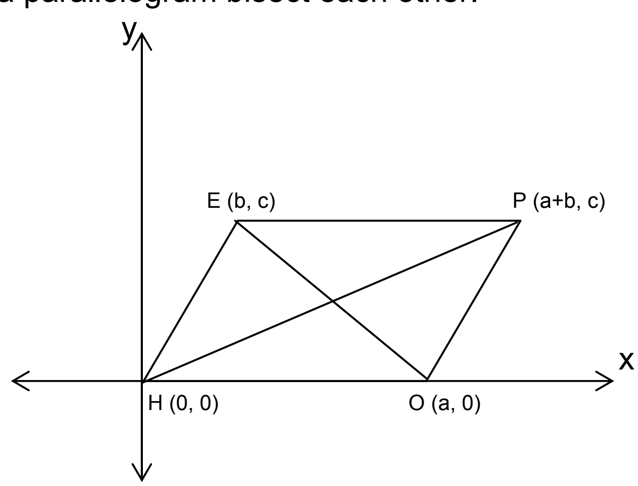
1. The medians to the legs of an isosceles triangle are equal.



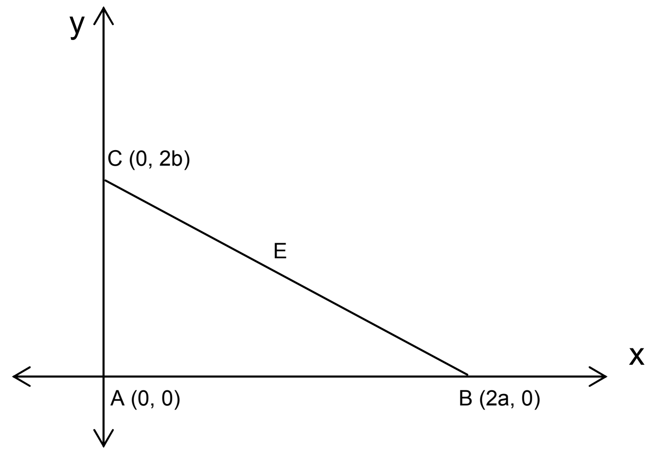
2. The diagonals of an isosceles trapezoid are equal.



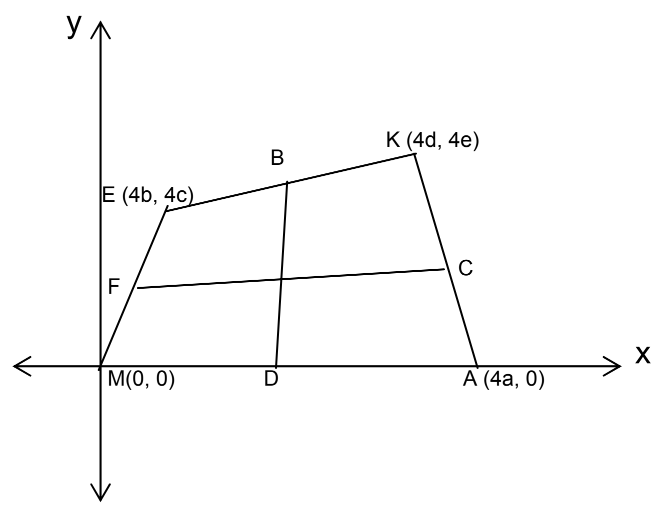
3. The diagonals of a parallelogram bisect each other.



4. The midpoint of the hypotenuse of a right triangle is equidistant from the vertices.



5. The segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.



Lesson 2

Circles on Coordinate Plane

You know from the previous chapter that the set of all points equidistant from a fixed point is called a ***circle***. The fixed point is called the center. If the circle is on the Cartesian coordinate plane, the distance formula will lead us to write an algebraic condition for a circle.

In the figure, if P has coordinates (x, y), using the distance formula will give us

$$\sqrt{(x-0)^2+(y-0)^2} = r$$

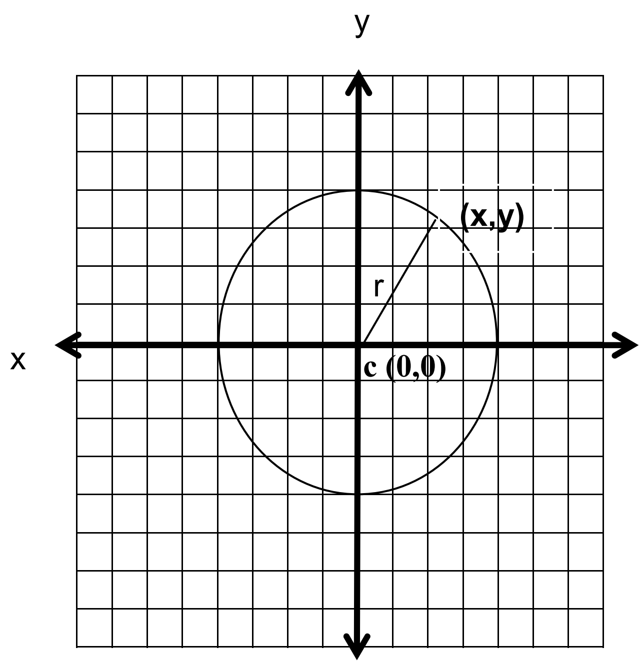
$$\sqrt{x^2+y^2} = r$$

Squaring both sides, we get

$$x^2 + y^2 = r^2$$

Thus the equation of a circle with center at (0, 0) and radius r in standard form is

$$x^2 + y^2 = r^2$$



Example 1. Find the equation of the circle whose center is the origin and the radius is

a. r = 3
b. r = 1
c. r = 5
d. r = $2\sqrt{2}$

Solution:

a. C(0, 0) r = 3
The equation of the circle is $x^2 + y^2 = 3^2$
$x^2 + y^2 = 9$

b. C(0, 0) r = 1
The equation of the circle is $x^2 + y^2 = 1^2$
$x^2 + y^2 = 1$

c. C(0, 0) r = 5
The equation of the circle is $x^2 + y^2 = 5^2$
$x^2 + y^2 = 25$

d. C(0, 0) r = $2\sqrt{2}$
The equation of the circle is $x^2 + y^2 = (2\sqrt{2})^2$
$x^2 + y^2 = 8$

Example 2. Determine the center and the radius of the circle whose equations are given.

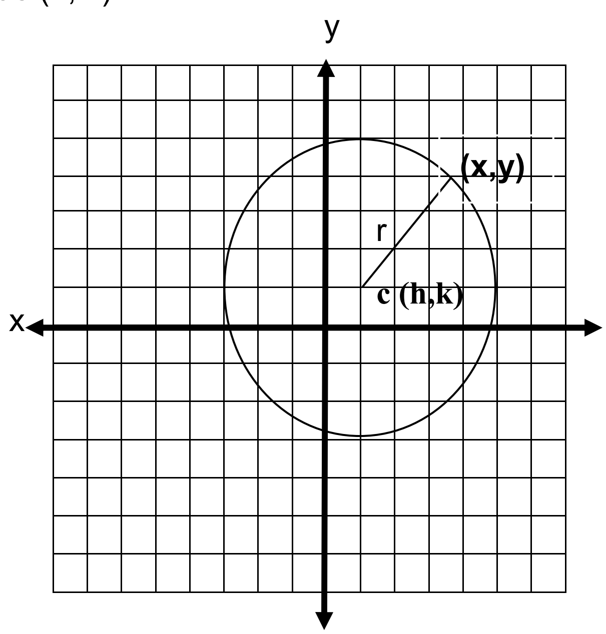
a. $x^2 + y^2 = 49$
b. $x^2 + y^2 = 36$
c. $x^2 + y^2 = 18$

Solutions:

The equations are of the form $x^2 + y^2 = r^2$, hence the center is the origin. The radius in each circle is

a. $r = 9$
b. $r = 6$
c. $r = 3\sqrt{2}$

Not all circles on the rectangular coordinate plane has its center at the origin. In the given circle below, its center is not the origin. We will represent the center of the circle which is not the origin as (h, k).



In the coordinate plane given above, center C has coordinates (h, k) and radius r. By Pythagorean Theorem, the distance from the center of the circle to a point A(x, y) on the circle can be solved. This will also give us the standard form of the equation of a circle.

$$AC = \sqrt{(x-h)^2 + (y-k)^2} = r$$
$$(x - h)^2 + (y - k)^2 = r^2$$

The standard form of the equation of the circle with center at (h, k) and radius ***r*** is

$$(x - h)^2 + (y - k)^2 = r^2$$

Example 1. Find the equation of a circle with center at (1, 5) and a radius of 3 units.

Solution: Substitute the following values in the standard form

$h = 1, \quad k = 5, \quad r = 3$

The equation is

$$(x - 1)^2 + (y - 5)^2 = 3^2$$
$$(x - 1)^2 + (y - 5)^2 = 9$$

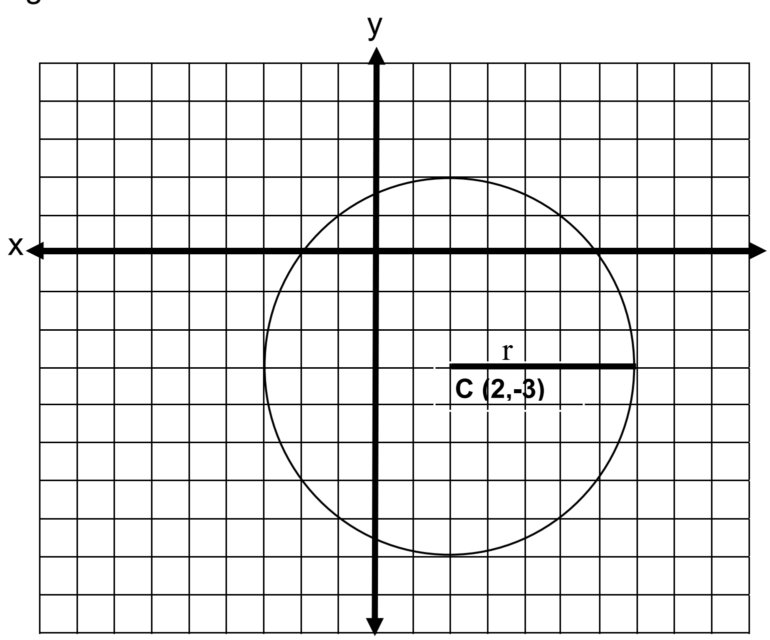
Example 2. Find the equation of a circle with center at (2, -3) and radius of 5 units.

Sketch the figure. $h = 2, \quad k = -3, \quad r = 5$

The equation is

$$(x - 2)^2 + (y + 3)^2 = 5^2$$
$$(x - 2)^2 + (y + 3)^2 = 25$$

The figure is



The standard form of the equation of a circle with center at C(h, k) and radius r can be presented in another form. This is done by squaring the binomials and simplifying the results.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2yk + k^2 = r^2$$
$$x^2 + y^2 - 2hx - 2yk + h^2 + k^2 - r^2 = 0$$

By assigning capital letters D, E and F to represent the constants, the equation will now assume this general form.

$$x^2 + y^2 + Dx + Ey + F = 0$$

Example 1. Find the radius and the center of the circle given its equation.

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

Solution:

First isolate the constant term at the right side of the equal sign by applying the addition property of equality

$$x^2 + y^2 - 4x - 6y - 12 = 0$$
$$x^2 + y^2 - 4x - 6y = 12$$

Then group the terms with x together and those with y together.

$$(x^2 - 4x) + (y^2 - 6y) = 12$$

Complete each group like in completing the square by adding the third term of the trinomial. Note that what you added to each group should be added to the right side of the equation also. (Application of addition property of equality)

$$(x^2 - 4x + \underline{4}) + (y^2 - 6y + \underline{9}) = 12 + 4 + 9$$
$$(x^2 - 4x + \underline{4}) + (y^2 - 6y + \underline{9}) = 25$$

Rewrite each perfect trinomial square into binomial factors.

$$(x - 2)^2 + (y - 3)^2 = 25$$
$$(x - 2)^2 + (y - 3)^2 = 5^2$$

Since the equation is in center-radius form, then we can determine the coordinates of the center and the radius of the circle.

The center is at (2, 3) and the radius is 5 units.

Example 2. Find the radius and the coordinates of the center of the circle given the Equation

$$x^2 + y^2 + 6x - 2y + 6 = 0$$

Solution:

Isolate first the constant term

$$x^2 + y^2 + 6x - 2y = -6$$

Then group the x and y together

$$(x^2 + 6x) + (y^2 - 2y) = -6$$

Add constants to each group by completing the square. Add to the right side of the equation what you will add to the left side.

$$(x^2 + 6y + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$$
$$(x^2 + 6y + 9) + (y^2 - 2y + 1) = 4$$

Write each trinomial as factors or square of binomial.

$$(x + 3)^2 + (y - 1)^2 = 4$$
$$x + 3)^2 + (y - 1)^2 = 2^2$$

The equation is in center-radius form. So the center of the circle is at (-3, 1) and its radius is 2 units.

Example 3. Tell whether the equation $x^2 + y^2 -4x + 8y + 24$ determines a circle.

Solution:

To solve this problem, you should be aware of these fact. The existence of a circle depends on the value of r^2.

In the standard form of equation of the circle, $(x - h)^2 + (y - k)^2 = r^2$, one of the following statements is always true if

$r^2 > 0$, the graph is a circle.
$r^2 = 0$, the graph is a point (We call this the point circle)
$r^2 < 0$, the graph or the circle does not exist

In the given example, solve for the value of r^2. In doing this, you simply follow the procedure in examples 1and 2.

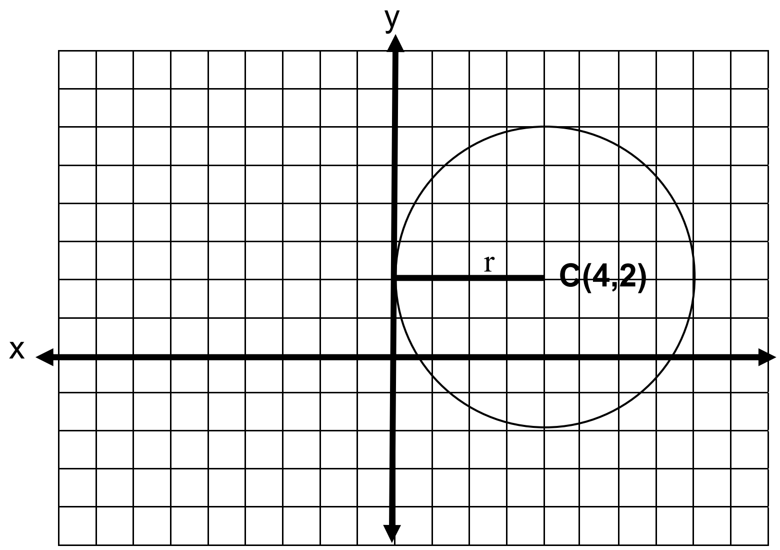
$$x^2 + y^2 - 4x + 8y + 24 = 0$$
$$x^2 + y^2 - 4x + 8y = - 24$$
$$(x^2 - 4x) + (y^2 + 8y) = - 24$$
$$(x^2 - 4x + 4) + (y^2 + 8y + 16) = -24 +4 + 16$$
$$(x - 2)^2 + (y + 4)^2 = -4$$

Since $r^2 = -4$, then the circle does not exist.

The following examples discuss of problems that involve circles in the coordinate plane. Each problem is treated differently according to what is given and what is being asked .

Example 4. Find the equation of a circle whose center is at (4, 2) and is tangent to y-axis.

Sketch the figure.



Solution:

Since the circle is tangent to y-axis, the radius of the circle is perpendicular to y-axis. It also means that the length of the radius is also the length of the perpendicular segment from the center of the circle to y –axis. From the figure, you can determine that the point of tangency is at (0, 2). To find the length of the radius, use the distance formula.

$$r = \sqrt{(4-0)^2 + (2-2)^2}$$

$$= \sqrt{4^2 + 0}$$

$$= \sqrt{16}$$

$$r = 4$$

To solve for the equation , use the coordinates of the center (4, 2) and the computed length of radius $r = 4$.

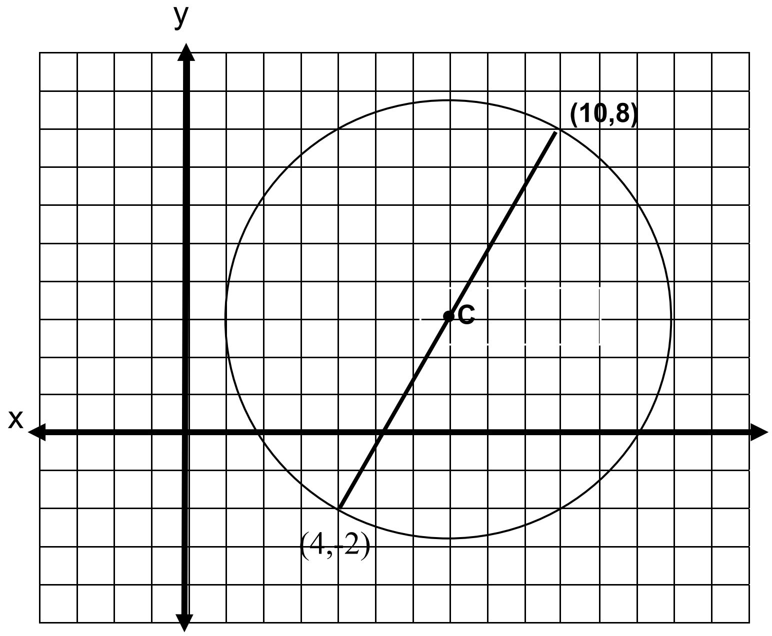
$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 4)^2 + (y - 2)^2 = 4^2$$

$$x^2 - 8x + 16 + y^2 - 4y + 4 = 16$$

$$x^2 + y^2 - 8x - 4y + 4 = 0$$

Example 5. Write the equation of the circle with the given condition. (10, 8) and (4, -2) are the endpoints of the diameter. Sketch the figure.



Solution:

In a circle, the radius is one-half of the diameter. Since the given are the endpoints of the diameter, then the center of the circle is the midpoint of the diameter.

$$M\left(\frac{10+4}{2},\frac{8+(-2)}{2}\right)$$

$$M\left(\frac{14}{2},\frac{6}{2}\right)$$

$$M(7,3)$$

The next step is to get the length of the radius. Since radius is one-half of the circle, so get the distance from the center to one of the endpoint of the diameter. Any endpoint will do.

$$r = \sqrt{(10-7)^2+(8-3)^2}$$

$$r = \sqrt{4^2+5^2}$$

$$r = \sqrt{16+25}$$

$$r = \sqrt{41}$$

To find the equation of the line, use C(7, 3) and r = $\sqrt{41}$

$$(x - 7)^2 + (y - 3)^2 = (\sqrt{41})^2$$
$$x^2 - 14x + 49 + y^2 - 6y + 9 = 41$$
$$x^2 + y^2 - 14x - 6y + 17 = 0$$

Example 6. Write the equation of the circle with center at (-8, 5) and passing through A(- 6, 4).

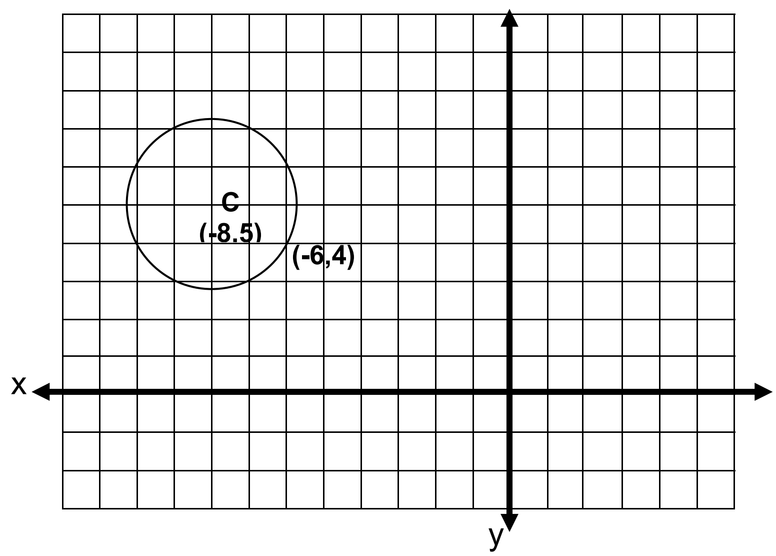
Solution: Since the circle is passing through A, then the distance from the center to A is the length of the radius of the circle. Compute first the radius of the circle.

$$r = \sqrt{[-8-(-6)]^2 + (5-4)^2}\ .$$

$$r = \sqrt{(-2)^2 + 1^2}$$

$$r = \sqrt{4+1}$$

$$r = \sqrt{5}$$



The equation of the circle is

$$(x + 8)^2 + (y - 5)^2 = (\sqrt{5})^2$$

$$x^2 + 16x + 64 + y^2 - 10y + 25 = 5$$

$$x^2 + y^2 + 16x - 10y + 84 = 0$$

Try this out.

A. Determine if the following are equations of a circle, a point or a circle that does not exist.

1. $x^2 + y^2 = 3$
2. $x^2 + y^2 - 12 = 0$
3. $x^2 + y^2 + 121 = 0$
4. $(x - 5)^2 + y^2 = 1$
5. $x^2 + y^2 - 10x - 8y + 41 = 0$

B. Give the center and the radius of each circle.

1. $x^2 + y^2 = 25$
2. $x^2 + y^2 - 12 = 0$
3. $x^2 + (y - 3)^2 = 121$

4. $(x - 7)^2 + (y - 5)^2 = 18$
5. $(x + 1)^2 + (y - 4)^2 = 3$
6. $(x - 8)^2 + y^2 = 49$
7. $\left(x + \frac{1}{2}\right)^2 + (y - 7)^2 = 25$
8. $x^2 + y^2 + 6x + 16y - 11 = 0$
9. $x^2 + y^2 + 2x - 6y - 8 = 0$
10. $x^2 + y^2 - 4x - 12y + 30 = 0$

C. Write the equation of a circle in standard form with center C and radius r given.

1. C(0, 0), $r = 4$
2. C(0, 0), $r = 2\sqrt{3}$
3. C(1, 1), $r = 3$
4. C(-2, -5), $r = \sqrt{10}$
5. C(-3, 4), $r = 2\sqrt{2}$
6. C(2, -5), $r = 5$
7. C(0, 6), $r = 6$
8. C(-4, 0), $r = 3.5$
9. C(0, -5), $r = \sqrt{5}$
10. C(3, 0), $r = 3\sqrt{3}$

D. Solve the following problems. Sketch the figure. Show the complete solution.
1. Write the equation of the circle with center at (3, -1) and tangent to the x-axis.
2. Write the equation of a circle with center at (2, 5) and passing through (-2, 1).
3. The line segment joining (-2, 5) and (-2, -3) is a diameter.
4. A circle is tangent to both axes and the radius at the first quadrant is 3 units.
5. A circle is tangent to the line $3x - 4y = 24$ and the center is at (1, 0).

Lets summarize

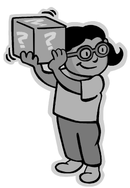
1. Proving theorems in geometry analytically is also known as coordinate proof.
2. In coordinate proof, the location of one of the vertices and one of the sides of the figure can help in proving easily the theorem.
3. The set of all points equidistant from a fixed point is called a circle. The fixed point is called the center.
4. The standard form of equation of a circle with center at the origin and radius r is
$$x^2 + y^2 = r^2$$
5. The standard form of equation of a circle with center at (h, k) and radius r is
$$(x - h)^2 + (y - k)^2 = r^2$$

6. The general form of equation of a circle is

$$x^2 + y^2 + Dx + Ey + F = 0$$

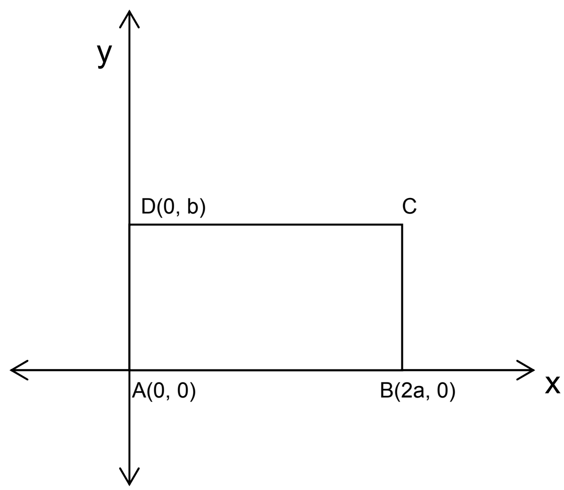
7. The existence of a circle depends upon the value of r^2.

If $r^2 > 0$, the circle exist.
If $r^2 = 0$, point circle exist.
If $r^2 < 0$, the circle does not exist.



What have you learned

1.



ABCD is a rectangle, What is the coordinate of C ?

2. In figure in # 1, find the length of $\overline{AC}$.
3. In doing coordinate proof, it is always simpler to put one of the vertices of the polygon or figure on the ___________.

4. What is the standard form of the equation of a circle if the center is the origin and the radius is r units?

What is the center and the radius of the following circles given their equations?

5. $x^2 + y^2 = 64$
6. $x^2 + (y + 1)^2 = 25$
7. $(x – 6)^2 + y^2 = 1$
8. $(x – 3)^2 + (y + 7)^2 = 12$
9. $(x – 1)^2 + (y +1)^2 = 49$

10. What is the equation of the circle whose center is at (-4, -1) and passing through the origin? Sketch the figure. Express the answer in general form.

Answer Key

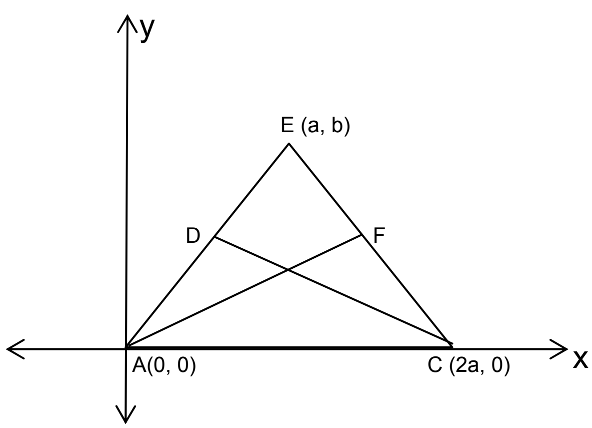
How much do you know.

1. c
2. a
3. b
4. C(0,0) or origin, r = 4
5. C(0, 0) or origin, r = 5
6. C(3, -1), r = 6
7. C(2, -5), r = $\sqrt{13}$
8. $(x - 2)^2 + (y + 1)^2 = 49$
9. $x^2 + y^2 + 4x + 8y - 20 = 0$
10. $x^2 + y^2 - 10y - 27 = 0$

Lesson 1

A. 1. A(a, b)
   2. B(a + c, b)
   3. C$\left(\frac{b}{2}, \frac{c}{2}\right)$
   4. C$\left(\frac{a}{2}, c\right)$

B. 1.



Given: ACE is an isosceles triangle.
$\overline{AF}$ and $\overline{DC}$ are medians

Prove: AF = DC

Solution:

1. First determine the coordinates of D and F.

Since $\overline{AF}$ and $\overline{DC}$ are medians, then D and F are midpoints.

$$D\left(\frac{a+0}{2},\frac{b+0}{2}\right)$$

$$D\left(\frac{a}{2},\frac{b}{2}\right)$$

$$F\left(\frac{a+2a}{2},\frac{b}{2}\right)$$

$$F\left(\frac{3a}{2},\frac{b}{2}\right)$$

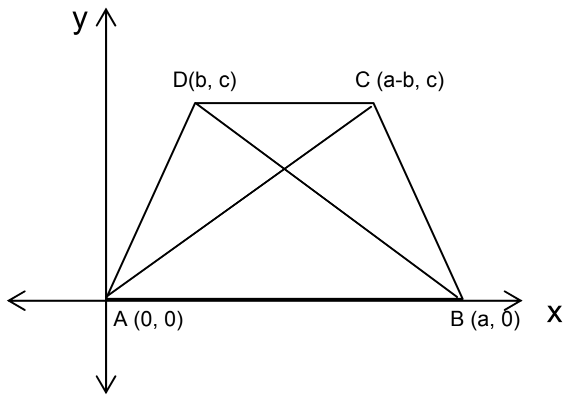
2. After finding the coordinates of D and F, determine the length of $\overline{AF}$ and $\overline{DC}$.

$$\begin{aligned}
AF &= \sqrt{\left(\frac{3a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2} \\
&= \sqrt{\left(\frac{3a}{2}\right)^2+\left(\frac{b}{2}\right)^2} \\
&= \sqrt{\frac{9a^2}{4}+\frac{b^2}{4}} \\
&= \sqrt{\frac{9a^2+b^2}{4}} \\
AF &= \frac{\sqrt{9a^2+b^2}}{2}
\end{aligned}$$

$$\begin{aligned}
DC &= \sqrt{\left(2a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2} \\
&= \sqrt{\left(\frac{3a}{2}\right)^2+\left(-\frac{b}{2}\right)^2} \\
&= \sqrt{\frac{9a^2}{4}+\frac{b^2}{4}} \\
&= \sqrt{\frac{9a^2+b^2}{4}} \\
DC &= \frac{\sqrt{9a^2+b^2}}{2}
\end{aligned}$$

Based on computations, AF = DC. Therefore, the two medians are equal and we can conclude that that the medians to the legs of an isosceles triangle are equal.

2.



Given: ABCD is an isosceles trapezoid.

$\overline{AC}$ and $\overline{BD}$ are its diagonals

Prove: $\overline{AC} \cong \overline{BD}$

Solution:

To prove that $\overline{AC} \cong \overline{BD}$ we have to show that AC = BD.

$$AC = \sqrt{(a-b-0)^2 + (c-0)^2}$$

$$= \sqrt{(a-b)^2 + c^2}$$

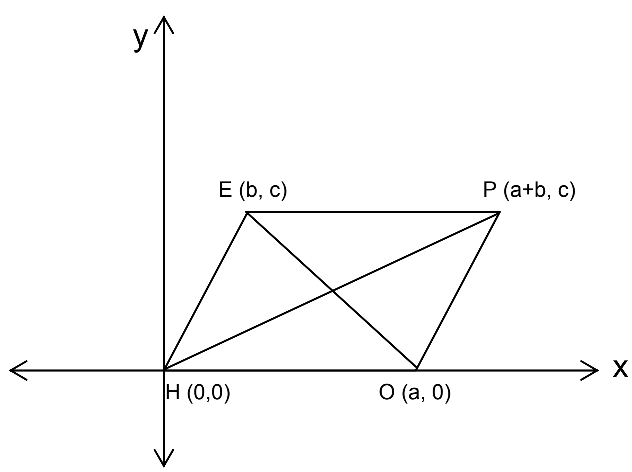
$$AC = \sqrt{a^2 - 2ab + b^2 + c^2}$$

$$BD = \sqrt{(a-b)^2 + (0-c)^2}$$

$$= \sqrt{a^2 - 2ab + b^2 + c^2}$$

The computations showed that AC = BD. Thus we can conclude that the diagonals of an isosceles trapezoid are congruent.

3.



Given: HOPE is a parallelogram
$\overline{HP}$ and $\overline{EO}$ are the diagonals

Prove: $\overline{HP}$ and $\overline{EO}$ bisect each other

Solution:

The simplest way of proving this is to show that the midpoints of the two diagonals are one and the same.

1. Find the midpoints of $\overline{HP}$ and $\overline{EO}$ and compare.

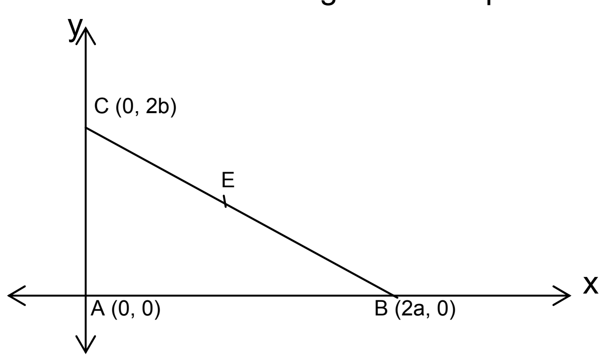
M (HP) $\left(\frac{a+b+0}{2},\frac{c}{2}\right)$

M (HP) $\left(\frac{a+b}{2},\frac{c}{2}\right)$

M (OE) $\left(\frac{a+b}{2},\frac{c}{2}\right)$

The results showed that $\overline{HP}$ and $\overline{EO}$ have the same midpoint. Hence they bisect each other. We can conclude that the diagonals of a parallelogram bisect each other.

4.



Given: ∆ABC is a right triangle. E is the midpoint of the hypotenuse BC.

Prove: CE = BE = AE

Solutions:

Since E is the midpoint of BC, then its coordinates are

$$E\left(\frac{2a}{2},\frac{2b}{2}\right)$$

E(a, b)

After finding the coordinates of E, find CE, BE and AE. Then compare the lengths.

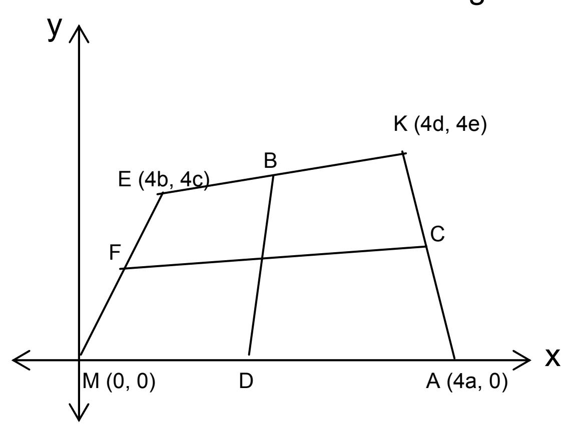
$$CE = \sqrt{(0-a)^2+(b-2b)^2}$$
$$= \sqrt{(-a)^2+(-b)^2}$$
$$CE = \sqrt{a^2+b^2}$$

$$BE = \sqrt{(2a-a)^2+b^2}$$
$$BE = \sqrt{a^2+b^2}$$

$$AE = \sqrt{(a-0)^2+(b-0)^2}$$
$$AE = \sqrt{a^2+b^2}$$

Since the three segments CE, BE and AE have the same lengths, we can therefore conclude that E is equidistant from the vertices of the right triangle.

5.



Given: MAKE is a quadrilateral
B, C, D and F are midpoints of the sides

Prove: $\overline{FC}$ and $\overline{BD}$ bisect each other

Solutions:

1. Get the coordinates of F, B, C and D.

$$F\left(\frac{4b+0}{2},\frac{4c+0}{2}\right)$$

F$\left(2b, 2c\right)$

B$\left(\frac{4b+4d}{2}, \frac{4c+4e}{2}\right)$

B$\left(2b+2d, 2c+2e\right)$

C$\left(\frac{4d+4a}{2}, \frac{4e}{2}\right)$
C$\left(2d+2a, 2e\right)$

D$\left(\frac{4a}{2}, \frac{0}{2}\right)$

D$\left(2a, 0\right)$

2. Get the midpoints of $\overline{FC}$ and $\overline{BD}$.

   Midpoint of $\overline{FC}$

   M$_1\left(\frac{2b+2d+2a}{2}, \frac{2c+2e}{2}\right)$
   M$_1\left(b+d+a, c+e\right)$

   Midpoint of $\overline{BD}$

   M$_2\left(\frac{2b+2d+2a}{2}, \frac{2c+2e+0}{2}\right)$

   M$_2\left(b+d+a, c+e\right)$

You can see that the midpoints of $\overline{FC}$ and $\overline{BD}$ are both (a+b+d, c+e). Therefore, we can conclude that the segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

Lesson 2

A.
1. a circle
2. a circle
3. the circle does not exist
4. a circle
5. a point circle

B.

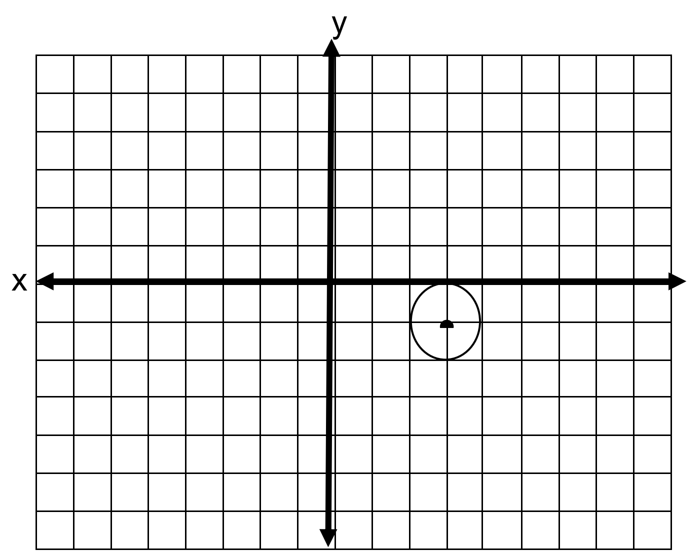
1. C(0, 0), r = 5
2. C(0, 0), r = $2\sqrt{3}$
3. C(0, 3), r = 11
4. C(7, 5), r = $3\sqrt{2}$
5. C(-1, 4), r = $\sqrt{3}$
6. C(8, 0), r = 7
7. C$\left(-\frac{1}{2}, 7\right)$, r = 5
8. C(-3, -4), r = 6
9. C(-1, 3), r = $3\sqrt{2}$
10. C(2, 6), r = $\sqrt{10}$

C.

1. $x^2 + y^2 = 16$
2. $x^2 + y^2 = 12$
3. $(x - 1)^2 + (y - 1)^2 = 9$
4. $(x + 2)^2 + (y + 5)^2 = 10$
5. $(x + 3)^2 + (y - 4)^2 = 8$
6. $(x - 2)^2 + (y + 5)^2 = 25$
7. $x^2 + (y - 6)^2 = 36$
8. $(x + 4)^2 + y^2 = 12.25$
9. $x^2 + (y + 5) = 5$
10. $(x - 3)^2 + y^2 = 27$

D.

1. C(3, -1) , tangent to x-axis



Solution:

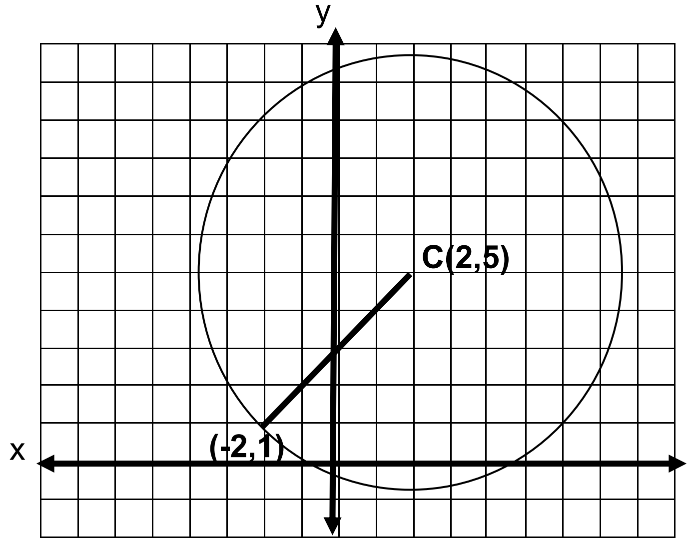
Since the circle is tangent to x-axis, then r is $\perp$ to x-axis. The distance from the center of the circle to the x-axis or the length of radius r is l unit.

The equation of the circle with center at (3, -1) and r = 1 is

$$(x - 3)^2 + (y + 1) = 1$$
$$x^2 - 6x + 9 + y^2 + 2y + 1 = 1$$
$$x^2 + y^2 - 6x + 2y + 10 - 1 = 0$$
$$x^2 + y^2 - 6x + 2y + 9 = 0$$

2. Center at (2, 5) and passing through (-2, 1)

Solution: Since the center is known what we need is the length of the radius. To find the length of the radius, use the other point as the other end of the radius.



r = $\sqrt{[2-(-2)]^2+(5-1)^2}$

r = $\sqrt{(2+2)^2+4^2}$

r = $\sqrt{4^2+4^2}$

r = $\sqrt{16+16}$

r = $\sqrt{32}$

r = $4\sqrt{2}$

The equation of the circle therefore is

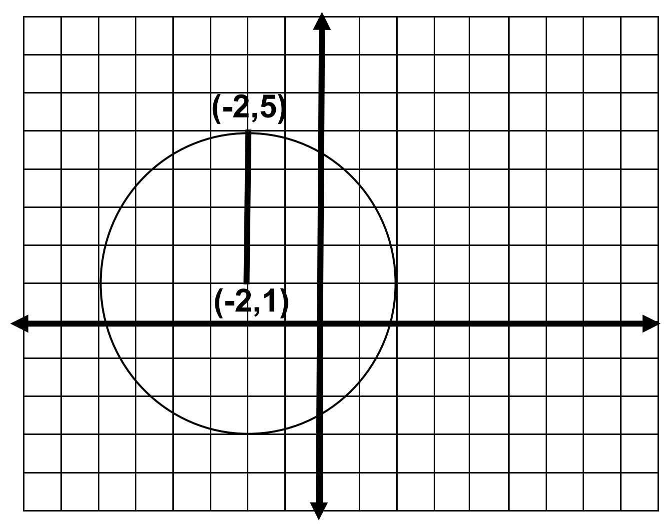
$(x-2)^2 + (y-5)^2 = (4\sqrt{2})^2$

$x^2 - 4x + 4 + y^2 - 10y + 25 = 32$

$x^2 + y^2 - 4x - 10y - 3 = 0$

3. Line segment joining (-2, 5) and (-2, -3) is a diameter.

Solution: The midpoint of the segment is the center of the circle. Find the coordinates of the midpoint.



M $\left(\frac{-2+(-2)}{2}, \frac{5-3}{2}\right)$

M $\left(\frac{-4}{2}, \frac{2}{2}\right)$

M (-2, 1)

Then find the length of the radius using the midpoint and one of the endpoints of the diameter .

r = $\sqrt{[-2-(-2)]^2+(5-1)^2}$

= $\sqrt{(-2+2)^2+4^2}$

= $\sqrt{0+4^2}$

= $\sqrt{4^2}$

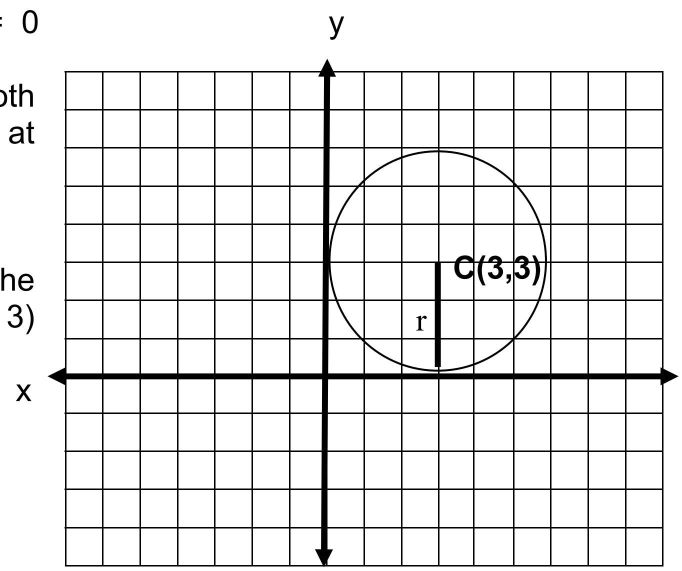
$r = 4$

The equation of the circle with center at (-2, 1) and r = 4 is

$$(x + 2)^2 + (y - 1)^2 = 4^2$$
$$x^2 + 4x + 4 + y^2 - 2y + 1 = 16$$
$$x^2 + y^2 + 4x - 2y - 11 = 0$$

4. Solution: Since the circle is tangent to both axes, therefore the center of the circle is at equal distance from both x and y axes.

That distance is 3 units since the radius is given at 3 units. The center is also at the first quadrant. The circle passes through (0, 3) and (3, 0). The center is at (3, 3)

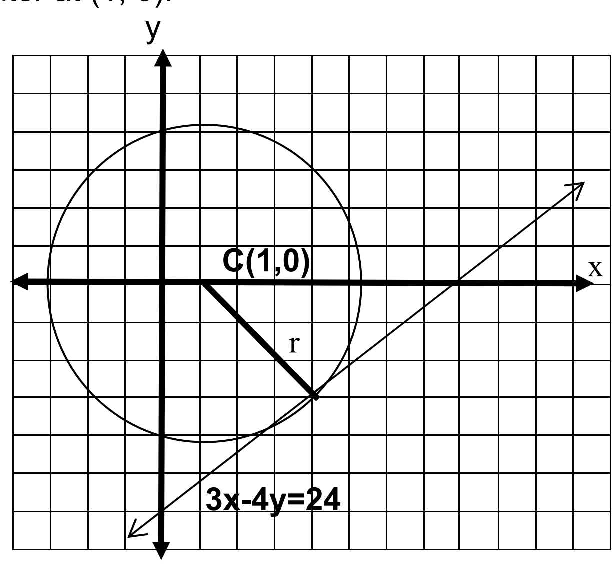


The equation of the circle is

$$(x - 3)^2 + (y - 3) = 3^2$$
$$x^2 - 6x + 9 + y^2 - 6y + 9 = 9$$
$$x^2 + y^2 - 6x - 6y + 9 = 0$$

5. Tangent to the line $3x - 4y = 24$ with center at (1, 0).

Solution: The radius of the circle is equal to the distance of the center (1, 0) from the line $3x - 4y = 24$.



To find the distance from a point to a line, we use the formula

$$d = \frac{Ax_1 + By_1 + C}{-\sqrt{A^2 + B^2}}$$

where A and B are the coefficients of x and y, and C is the constant in the equation of line. x_1 and y_1 are the coordinates of the point.

Therefore in the given,
A = 3, B = -4 and C = -24. $x_1 = 1$, $y_2 = 0$

$$r = \frac{3(1) + (-4)(0) - 24}{-\sqrt{3^2 + 4^2}}$$

$$= \frac{3+0-24}{-\sqrt{9+16}}$$

$$= \frac{-21}{-5}$$

$$r = \frac{21}{5}$$

The equation of the circle with center at (1, 0) and radius $r = \frac{21}{5}$ is

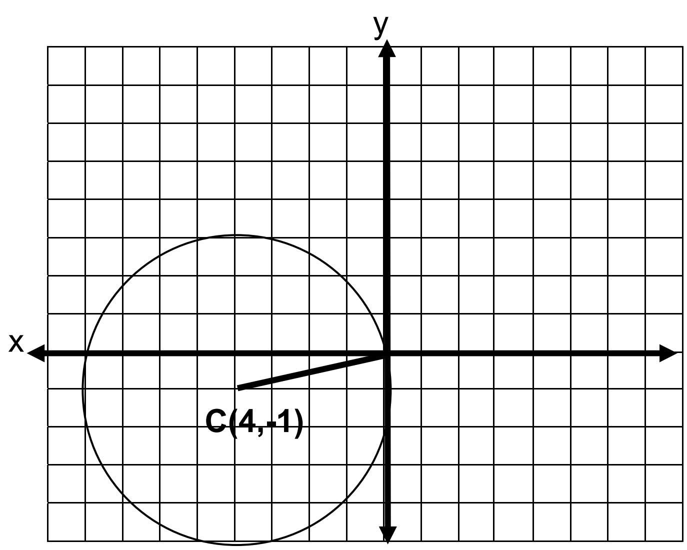
$$(x-1)^2 + y^2 = \left(\frac{21}{5}\right)^2 = \frac{441}{25}$$

What have you learned

1. C(2a, b)
2. AC = $\sqrt{2a^2 + b^2}$
3. origin
4. $x^2 + y^2 = r^2$
5. C(0, 0), $r = 8$
6. C(0, -1), $r = 5$
7. C(6, 0), $r = 1$
8. C(3, -7), $r = 2\sqrt{3}$
9. C(1, -1), $r = 7$

10. Solution: Find the length of r using the origin and the coordinates of the center (-4, -1).

$$r = \sqrt{(-4+0)^2 + (-1-0)^2}$$
$$= \sqrt{(-4)^2 + (-1)^2}$$
$$= \sqrt{16+1}$$
$$= \sqrt{17}$$



The equation is

$$(x+4)^2 + (y+1)^2 = (\sqrt{17})^2$$
$$x^2 + 8x + 16 + y^2 + 2y + 1 = 17$$
$$x^2 + y^2 + 8x + 2y + 17 - 17 = 0$$
$$x^2 + y^2 + 8x + 2y = 0$$