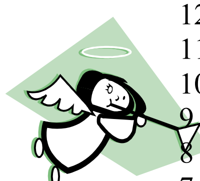
BUREAU OF SECONDARY EDUCATION
DEPARTMENT OF EDUCATION

DISTANCE LEARNING MODULE
MATHEMATICS 2

SEQUENCES AND SERIES

UNIT VII

The number of gifts sent in the popular Christmas Carol "12 days of Christmas" form a sequence. A part of the song goes this way "On the 12th day of Christmas my true love gave to me

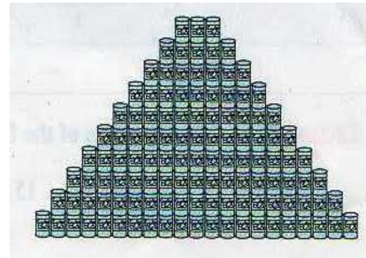


12 drummers drumming
11 pipers piping
10 lords a leaping
9 ladies dancing
8 maids a milking
7 swans a swimming
6 geese a laying
5 golden rings
4 calling birds
3 French hens
2 turtle doves, and
a partridge in a pear tree."

How many gifts were given by the true love on the 12th day of Christmas?

. The answer is 78. How did I solve it? You will find out as you learn the concepts in one of the lessons on arithmetic series. In this unit you will learn the different sequences and series and how they are applied in real life

Lesson 1 Sequences



A supermarket displays canned goods by stacking them, so that there are 10 rows of cans with 3 cans in the top row. If each row, below the top row, had two more cans than the rows just above it, how many cans could there be at the bottom row?

Starting from the top row, the number of cans in each row can be listed as follows:

3, 5, 7, 9, 11, 13, 15, 17, 19, 21

The stacks of cans are arranged in some order such that there are two cans more below each row. Any such ordered arrangement of a set of numbers is called a *sequence*.

The list of numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 is called a sequence. Each of the numbers of a sequence is called a *term* of the sequence. The first term in the sequence is 3, the second term is 5, while the third term is 7 and the 10^{th} term is 21.

Sequences are classified as finite and infinite.

A finite sequence contains a finite number of terms.

Examples:

1, 1, 2, 3, 5, 8
1, 2, 3, 4, 5, …, 8
1, -1, 1, -1

An infinite sequence contains an infinite number of terms. The ellipsis, "…" is often used to show that a sequence is infinite

Examples:

1, 3, 5, 7, …
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
1, 1, 2, 3, 5, 8, …

> A ***sequence*** is a set of numbers written in a specific order:
>
> $$a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_n.$$
>
> The number a_1 is called the 1^{st} term, a_2 is the 2^{nd} term, and in general, a_n is the nth term.

You can easily find the next term in a sequence by simply discovering a pattern as to how the terms are formed. You will find that either a constant number is added or subtracted or multiplied or divided to get the next term or a certain series of operations is performed to get the next term.

Examples:

Find the next term in each sequence.

1. 17, 22, 27, 32, …

Notice that 5 is added to get the next terms or numbers in the sequence.

17	1^{st} term
17 + 5 = 22	2^{nd} term
22 + 5 = 27	3^{rd} term
27 + 5 = 32	4^{th} term

What could be the next term?

2. $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11} \ldots$

The fractions have 1 as a common numerator. The denominators 2, 5, 8, 11 form a sequence, by adding 3 to the preceding numbers (2 precedes 5, 5 precedes 8, 8 precedes 11).

What could be next term after $\frac{1}{11}$?

3. 5, 10, 20, 40,…

For this example, 2 is multiplied by 5 to get 10, 2 is multiplied by 10 to get 20 and 2 is also multiplied by 20 to get 40. So the next term is 80, the result of multiplying 40 by 2.

4. 4, -4, 4, -4,…

In this example, it is easy to guess that the next term is 4 since the terms in the sequence alternately are positive and negative 4. Actually, the first, second, and third terms were multiplied by -1 to get the second, third and fourth terms, respectively.

Let's Practice for Mastery 1:

A. Write **F** if the sequence is finite or **I** if the sequence is infinite.

1. 2, 3, 4, 5, ….., 10
2. 7, 10, 13, 16, 19, 22, 25
3. 4, 9, 14, 19, …
4. 1, 4, 9, 16, 25, …., 144
5. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}$

Let's Do It 1:

Why are Policemen Strong?

Policemen Have 'Arresting' Personalities

Find the next number in each sequence. Replace it with the letters on the left of each sequence. Write the letters that corresponds to the sequence on the box below to decode the answer to the puzzle. (Source: Math Journal)

A 2, 5, 11, 23, __
B 2, 4, 16, __
C 7, 13, 19, __
D 19, 16, 13, __
E 4, 8, 20, 56, __
F 2, 2, 4, 6, 10, 16, __
H 1, 1, 2, 4, 7, 13, __
I 3, 6, 12, 24, __
L 10, 11, 9, 12, 8, __
N 2, 6, 18, 54, __
O 20, 19, 17, __
P 2, 3, 5, 7, 9, 11, 13, 15, __
R 13, 26, 39, __
S 5, 7, 13, 31, __
T 1, 1, 2, 4, 7, 13, 24, __
U 1, 1, 1, 2, 3, 4, 6, 9, 13, __
Y 1, 2, 2, 4, 3, 6, 4, 8, 5, 10, __

256	164	25	47	19	85	164		44	24	164	6		25	47	162

24	14	13	10		19	17		44	52	47	26	26	48	25

Let's Check Your Understanding 1:

A. Write **F** if the sequence is finite or **I** if the sequence is infinite.

1. 2, 6, 18, 54
2. 3, 9, 27, 81, …., 729, …
3. -2, 4, -8, 16, …..
4. 100, 97, 94, 91, …, -2
5. $\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{32},\frac{1}{64}$

B. Find the next term in the sequence of numbers in A.

Lesson 2 The Terms of a Sequence

Frequently, a sequence has a definite pattern that can be expressed by a rule or formula. In the simple sequence

2, 4, 6, 8, 10, ….

each term is paired with a natural number by the rule $a_n = 2n$.

Notice how the formula $a_n = 2n$ gives all the terms of the sequence. Substitute 1, 2, 3, and 4 for n in $a_n = 2n$:

$a_n = 2n$	$a_n = 2n$	$a_n = 2n$	$a_n = 2n$
$a_1 = 2(1) = 2$	$a_2 = 2(2) = 4$	$a_3 = 2(3) = 6$	$a_4 = 2(4) = 8$

Using this formula, can you find a_{103} or the 103^{rd} term of this sequence?

Examples:

1. Find the first four terms of the sequence whose general term is given by
$a_n = 2n – 1$.

Solution:

To find the first, second, third and fourth terms of this sequence, simply substitute 1, 2, 3, 4 for n in the formula $a_n = 2n-1$.

If the general term is $a_n = 2n – 1$, then the

1^{st} term is $a_1 = 2(1) – 1 = 1$
2^{nd} term is $a_2 = 2(2) – 1 = 3$
3^{rd} term is $a_3 = 2(3) – 1 = 5$
4^{th} term is $a_4 = 2(4) – 1 = 7$.

The first four terms of this sequence are the odd numbers 1, 3, 5, and 7. The whole sequence can be written as

$$1, 3, 5, \ldots, 2_n – 1$$

2. Write the first four terms of the sequence defined by $a_n = \frac{1}{n+1}$.

Solution:

Replacing n with 1, 2, 3, and 4, respectively the first four terms are:

1^{st} term : $a_1 = \frac{1}{1+1} = \frac{1}{2}$

2^{nd} term : $a_2 = \frac{1}{2+1} = \frac{1}{3}$

3rd term : $a_3 = \frac{1}{3+1} = \frac{1}{4}$

4th term: $a_4 = \frac{1}{4+1} = \frac{1}{5}$

The sequence defined by $a_n = \frac{1}{n+1}$ can now be written as

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots, \frac{1}{n+1}.$$

3. Find the 1st 5 terms of the sequence defined by $a_n = \frac{(-1)^n}{2^n}$.

Solution:

Again by simple substitution,

1st term : $a_1 = \frac{(-1)^1}{2^1} = -\frac{1}{2}$

2nd term: $a_2 = \frac{(-1)^2}{2^2} = \frac{1}{4}$

3rd term: $a_3 = \frac{(-1)^3}{2^3} = -\frac{1}{8}$

4th term: $a_4 = \frac{(-1)^4}{2^4} = \frac{1}{16}$

5th term: $a_5 = \frac{(-1)^5}{2^5} = -\frac{1}{32}$

The sequence defined by $a_n = \frac{(-1)^n}{2^n}$ can be written as

$$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots, \frac{(-1)^n}{2^n}$$

Notice that the presence of (-1) in the sequence has the effect of making successive terms alternately negative and positive.

Let's Practice for Mastery 2:

A. Write the first four terms of the sequence whose nth term is given by the formula.

1. $a_n = n + 1$
2. $a_n = 3n - 1$
3. $a_n = 2^n$
4. $a_n = n^2 + 1$
5. $a_n = \frac{n}{n+1}$

B. Find the indicated term of the sequence whose nth term is given by the formula.

1. $a_n = 3n + 4$ $\quad a_{12}$
2. $a_n = 2 – 2n$ $\quad a_8$
3. $a_n = n(n - 1)$ $\quad a_{11}$
4. $a_n = (-1)^{n-1} n^2$ $\quad a_{15}$
5. $a_n = (\frac{1}{2})^n$ $\quad a_8$

Let's Check Your Understanding 2:

A. Write the first four terms of the sequence whose nth term is given by the formula.

1. $a_n = n – 1$
2. $a_n = n^2 - 1$
3. $a_n = 3^n$
4. $a_n = 1 - 2n$
5. $a_n = n^2 - \frac{1}{n}$

B. Find the indicated term of the sequence whose nth term is given by the formula.

1. $a_n = 2n - 5$ $\quad a_{10}$
2. $a_n = 2n + 1$ $\quad a_5$
3. $a_n = \frac{n}{n+1}$ $\quad a_{12}$
4. $a_n = (\frac{2}{3})^n$ $\quad a_5$
5. $a_n = (n + 2)(n + 3)$ $\quad a_{17}$

Lesson 3 Finding the nth Term of a Sequence

In Lesson 2, some terms of a sequence were found after being given the general term. In this lesson, the reverse is done. That is, given some terms of the sequence, we want to find an expression for the nth term.

Examples:

1. Find a formula for the nth term of the sequence 2, 8, 18, 32,…

Solution:

Solving a problem like this involves some guessing. Notice that the first four terms is twice a perfect square:

$$2 = 2(1)$$
$$8 = 2(4)$$
$$18 = 2(9)$$
$$32 = 2(16)$$

By writing each sequence with an exponent of 2, the formula for the nth term is:

$a_1 = 2 = 2(1)^2$
$a_2 = 8 = 2(2)^2$
$a_3 = 18 = 2(3)^2$
$a_4 = 32 = 2(4)^2$
.
.
.
$a_n = 2(n)^2 = 2n^2$

The formula of the nth term of the sequence 2, 8, 18, 32,… is $a_n = 2n^2$.

2. Find an expression for the nth term of the sequence 2, $\frac{3}{8}, \frac{4}{27}, \frac{5}{64}, \ldots$

Solution:

The first term can be written as $\frac{2}{1}$. The denominators 1, 8, 27, 64 are all perfect cubes, while the numerators are 1 more than the base of the cubes of the denominators:

$$a_1 = \frac{2}{1} = \frac{1+1}{1^3}$$

$$a_2 = \frac{3}{8} = \frac{2+1}{2^3}$$

$$a_3 = \frac{4}{27} = \frac{3+1}{3^2}$$

$$a_4 = \frac{5}{64} = \frac{4+1}{4^3}$$

Observing this pattern, the expression for the nth term is $a_n = \frac{n+1}{n^3}$.

3. Find the nth term of the sequence -2, 4, -8, 16, -32,…

Solution:

These numbers are powers of 2 and they alternate in sign.

The expression for the nth term is given by $a_n = (-1)^n 2^n$.

4. In $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \ldots$, examine how the numerators and denominators change as separate sequences. Having the knowledge of identifying the rule for the nth term, the expression now gives us $a_n = \frac{n+1}{n^2}$.

Not all sequences can be defined by a formula, like for the sequence of prime numbers.

Let's Practice for Mastery 3:

A. Find an expression for the nth term for each of the following sequences:

1. 3, 6, 9, 12,…
2. 3, 12, 27, 48,…
3. -2, 4, -8, 16,…
4. 4, 9, 14, 19,…
5. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ …
6. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}$ …

B. Find the seventh term in each sequence.

1. 2, 4, 6, 8, …
2. 100, 250, 400, …
3. 3, 10, 17, …
4. 1, 4, 9, 16, …
5. 7, 17, 27, 37, …

Let's Check Your Understanding 3:

A. Find an expression for the nth term for each of the following sequences:

1. 7, 10, 13, 16,…
2. 4, 8, 12, 16, 20,…
3. 1, 4, 9, 16,…
4 2, 16, 54, 128,…
5. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$…
6. $\frac{1}{4}, \frac{2}{10}, \frac{3}{28}, \frac{1}{32}$

B. Find the seventh terms in each sequence.

1. 0, 1, 1, 2, 3, …
2. 1, 3, 6, 10, …
3. 1, 2, 4, 8, 16, …
4. -1, 2, -3, 4, …
5. 300, 200, 100, 0, …

Lesson 4 Arithmetic Sequences

Look at the following sequences.

1. 4, 7, 10, 13,…
2. 33, 38, 43, 48,…
3. -2, -6, -10, -14,…
4. 100, 98, 96, 94,…
5. $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, …

Can you give the next two terms of each sequence above? How did you get the next terms in each case?

If you get the next two terms and the number added to the preceding terms to get the next terms then you are correct:

1. next two terms: 16, 19	the number added: 3
2. next two terms: 53, 58	the number added: 5
3. next two terms: -18, -22	the number added:-4
4. next two terms: 92, 90	the number added: -2
5. next two terms: $2\frac{1}{2}$, 3	the number added:½

Notice that a fixed number is **added** to the preceding term to get the next term in the sequences. These sequences are called ***arithmetic sequences.*** The fixed number added is called the *common difference d.*

> A sequence where each succeeding term is obtained by adding a fixed number is called an *arithmetic sequence*. The fixed number is called the *common difference d.*

In order to identify if a pattern is an arithmetic sequence we must examine consecutive terms. If all consecutive terms have a *common difference* you can conclude that the sequence is arithmetic.

Examples:

Consider the sequence of numbers. Find the next four terms of each.

1. 5, 25, 45, 65, …
2. 0, 9, 18, 27, …
3. 1, 9, 17, 25, …
4. -9, -4, 1, 6, …

In each sequence, how will you get the next terms?

First, find the common difference by subtracting the 2nd term from the 1st, the 3rd from the 2nd and so on. In symbols, $d = a_n - a_{n-1}$.

1. 5, 25, 45, 65, …

   25 – 5 = 20; 45 – 25 = 20; 65 – 45 = 20

   The common difference is 20. The fixed number 20 is added to the preceding terms to get the succeeding terms.

2. 0, 9, 18, 27, …

   9 – 0 = 9; 18 – 9 = 9; 27 – 18 = 9

   The common difference is 9.

3. 1, 9, 17, 25, …

9 – 1 = 8; 17 – 9 = 8; 25 – 18 = 8

The common difference is 8.

4. -9, -4, 1, 6, …

-4 – (-9) = -4 + 9 = 5; 1 – (-4) = 1 + 4 = 5; 6 – 1 = 5

The common difference is 5.

Now, you are ready to find the next four terms being asked for in the given sequence. See if you got these answers.

1. 5, 25, 45, 65, 85, 105, 125, 145
2. 0, 9, 18, 27, 36, 45, 54, 63
3. 1, 9, 17, 25, 33, 41, 49, 57
4. -9, -4, 1, 6, 11, 16, 21, 26

How many correct answers did you get? How did you get your answers?

Let's Practice for Mastery 4.1:

Determine whether the sequence is arithmetic or not. If it is, find the common difference and the next three terms.

1. 2, 5, 8, 11,…
2. 2, -4, 6, -8, 10,…
3. -6, -10, -14, -18,…
4. 40, 42, 44, 46,…
5. 1.2, 1.8, 2.4,…
6. 1, 5, 9, 13,…
7. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$
8. $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, ...$
9. 98, 95, 92, 89,…
10. $1, \frac{4}{3}, \frac{5}{3}, 2, …$

Let's Check Your Understanding 4.1:

Find the common difference and the next three terms of the given arithmetic sequence.

1. 1, 10, 19, 28,…
2. 5.5, 7, 8.5, 10,…
3. $\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, 4\sqrt{3}, …$
4. 1, 3, 5, 7,…
5. 43, 39, 35, …
6. 25, 34, 43, 52, …

Can you give the 100^{th} term of the arithmetic sequence -8, -3, 2, 7, …?

Knowing the common difference you can, but it would not be easy listing the sequence of numbers from -8, our first term (a_1) to the 100^{th} term or a_{100}.

You need to use an equation to define the nth term of the arithmetic sequence.

> Any *arithmetic sequence* is defined by the equation
>
> $$a_n = a_1 + (n - 1)d,$$
>
> where, a_n is the nth term, a_1 is the 1st term and d is the common difference.

Examples:

1. Find the 5th term of the arithmetic sequence for which the first term is 9 and the common difference is 7?

    a. Since d = 7 and $a_1 = 9$, we can easily find the next 5 terms of the arithmetic sequence as:

    9, 16, 23, 30, <u>37</u>

    b. Given: $a_1 = 9$, n = 5, and d = 7
    Find: a_5

    Substitute the given values in the formula:

    $$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_5 &= 9 + (5 - 1)7 \\ &= 9 + (4)7 \\ &= 9 + 28 \\ a_5 &= 37 \end{aligned}$$

    Therefore, 37 is the 5th term or a_5 of the sequence.

2. Find the 100th term of the sequence in #1.

    Given: $a_1 = 9$, n = 100, d = 7
    Find: a_{100}

    Substitute the given in the equation:

    $$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_{100} &= 9 + (100 - 1)7 \\ &= 9 + (99)7 \\ &= 9 + 693 \\ a_{100} &= 702 \end{aligned}$$

    Therefore, the 100th term or a_{100} is 702.

3. In the arithmetic sequence 1, 8, 15, 22, …, which term equals 519?

    Given: $a_n = 519$, $t_1 = 1$, d = 7
    Find: n

Substitute the given values in the formula:

$$a_n = a_1 + (n - 1)d$$
$$519 = 1 + (n - 1)7$$
$$519 = 1 + 7n - 7$$
$$519 = 7n - 6$$
$$519 + 6 = 7n$$
$$525 = 7n$$
$$\frac{525}{7} = n$$
$$n = 75$$

Therefore, the 75^{th} term or a_{75} of the arithmetic sequence is 519.

Let's Practice for Mastery 4.2:

Find the term indicated in each of the following arithmetic sequences.

1. 2, 4, 6, … 15^{th} term
2. 13, 16, 19, 22, … 25^{th} term
3. 99, 88, 77, 66, … 18^{th} term
4. 99, 87, 75, 63, … 12^{th} term
5. $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \ldots$ 20^{th} term

Let's Check Your Understanding 4.2:

Find the term indicated in each of the following arithmetic sequences.
1. -8, -3, 2, 7, … 23^{rd} term
2. 91, 84, 77, 70,… 17^{th} term
3. $\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, 4\sqrt{3}, \ldots$ 14^{th} term
4. 25, 34, 43, 52, … 10^{th} term
5. 10, 4, -2, -8, … 22^{nd} term

Lesson 5 Applications of Arithmetic Sequence in Real Life

Without our knowing it, we are applying the concept of arithmetic sequence in real life. The examples below illustrate some of these applications.

1. Mrs. Lacson bought a house and lot at the beginning of 1995 for Php1,500,000. If its value increased by Php100,000 each year, how much was it worth at the end of 2005?

   In 1995 the amount of the house and lot bought by Mrs. See was Php1,500,000. In the following year, 1996, Php100,000 was added to the original amount, thus having the new value of Php1,600,000.00. Let us tabulate to solve the problem.

Year	*# of years after purchase date*	*Value of the House and Lot*
1995	1st	1,500,0000
1996	2nd	1,500,0000 + 100,000 = 1,600,000
1997	3rd	1,600,0000 + 100,000 = 1,700,000
1998	4th	1,700,0000 + 100,000 = 1,800,000
1999	5th	1,800,0000 + 100,000 = 1,900,000
2000	6th	1,900,0000 + 100,000 = 2,000,000
2001	7th	2,000,0000 + 100,000 = 2,100,000
2002	8th	2,100,0000 + 100,000 = 2,200,000
2003	9th	2,200,0000 + 100,000 = 2,300,000
2004	10th	2,300,0000 + 100,000 = 2,400,000
2005	11th	2,400,0000 + 100,000 = 2,500,000

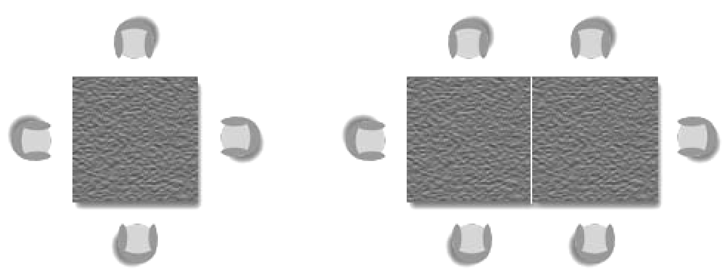
The house and lot is worth Php2,500,000 at the end of 2005.

Using the formula $a_n = a_1 + (n - 1)d$, we can solve the problem as shown below.

$a_{11} = 1{,}500{,}000 + (11 - 1)100{,}000$
$\quad = 1{,}500{,}000 + 1{,}000{,}000$
$a_{11} = 2{,}500{,}000$

Note that the resulting amount is the same as that in the table.

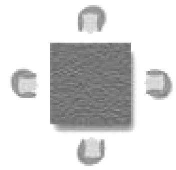
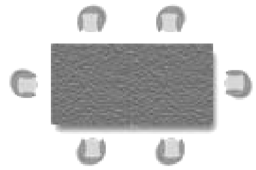
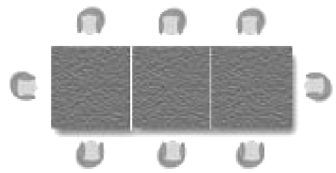
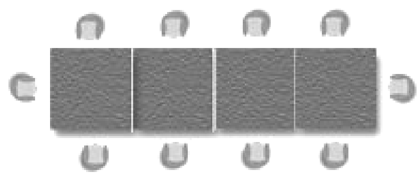
2. A restaurant has square tables which seat four people. When two tables are placed together, six people can be seated as illustrated below.



If 20 square tables are placed together to form one long table, how many people can be seated?

If 100 square tables are placed together to form one very long table, how many people can be seated?

You may use a table in order to see if there is a *pattern* that relates the number of tables to the number of people that can be seated.

Number of Tables placed together	**Diagram**	**Number of Seats**
1		4
2		6
3		8
4		10

The number of seats in the sequence begin with 4, 6, 8, 10, ...

To find the number *n* of people that can be seated at 20 tables that are placed together, we use the formula in finding the nth term of an arithmetic sequence, that is, $a_n = a_1 + (n - 1)d$.

Given: $a_1 = 4$, $d = 2$, $n = 20$
Find: a_{20}

Substitute the given values in the formula:

$$
\begin{aligned}
a_n &= a_1 + (n - 1)d \\
a_{20} &= 4 + (20 - 1)2 \\
&= 4 + (19)2 \\
&= 4 + 38 \\
a_{20} &= 42
\end{aligned}
$$

Therefore, 42 people could sit at 20 tables.

You can find the number of people that can be seated at 100 tables, using the same formula.

Given: $a_1 = 4$, $d = 2$, $n = 100$.
Find: a_{100}

Substitute the given in the formula:

$$a_n = a_1 + (n - 1)d$$
$$a_{100} = 4 + (100 - 1)2$$
$$= 4 + (99)2$$
$$= 4 + 198$$
$$a_{100} = 202$$

Therefore, 202 people could sit at 100 tables.

Let's Practice for Mastery 5:

Solve the following problems.

1. The force of gravity causes a ball to fall 16.1 decimeters during the first second, 48.3 the next second, 80.5 the third, and so on. How far will the body fall in 10 seconds?

2. SEJ Company offers you a job with choice of two salary increase plans. With Plan A, you receive an annual salary of Php100,000.00 and an annual increase of Php1000.00. With Plan B, you will receive a semiannual salary of Php50,000.00 and a semiannual increase of Php250. Which plan should you choose? How much money will you receive over five years for each of the plans?

3. In a Math competition, 10 questions were asked for each participant. For every wrong answer, a competitor loses $\frac{1}{2}$ point. If Lina earned 6 points during the first question, and then loses $\frac{1}{2}$ point for the 9 remaining questions. What was her score after the 10^{th} question?

Let's Check Your Understanding 5:

Customers paying an electric bill were given tickets with a number. The first customer got number 0214, the second customer got 0216, the third customer got 0218, and so on. The numbers on the ticket forms an arithmetic sequence.

a. If the sequence continues, what number will the 10^{th} customer get?

b. What is a_{20}?

c. Which customer got ticket number 0400?

d. If the last customer got ticket number 1664, how many customers entered the electric company?

Lesson 6 Finding the 1st Term and the Common Difference Given Two terms of Arithmetic Sequence

Examples:

1. The 3rd term of an arithmetic sequence is 8 and 7th term is 20, find the first term.

We can solve this in a simpler way. The sequence looks like the ones below.

_ , _ , 8 , _, _ , _, 20
a_1 a_3 a_7

If we consider 8 as our first term and 20 as our nth term, we can use our formula:

$$a_n = a_1 + (n - 1)d$$

Using $a_n = 20$, $a_1 = 8$ and n = 5 (since there are 5 terms from 8 to 20), then

$$a_n = a_1 + (n - 1)d$$
$$20 = 8 + (5 - 1)d$$
$$20 = 8 + 4d$$
$$20 - 8 = 4d$$
$$12 = 4d$$
$$\frac{12}{4} = d$$
$$d = 3$$

We can now use the value of d to solve for a_1 in the original problem. Substituting $a_7 = 20$ and d = 3 in the formula, we obtain

$$a_n = a_1 + (n - 1)d$$
$$20 = a_1 + (7 - 1)(3)$$
$$20 = a_1 + 6(3)$$
$$20 = a_1 + 18$$
$$20 - 18 = a_1$$
$$2 = a_1 \text{ or}$$
$$a_1 = 2$$

Let's Practice for Mastery 6:

The 10th and 12th terms of an arithmetic sequence are 11 and 14, respectively.

a. What is d?
b. What is a_1?
c. What is a_{234}?
d. How many terms are negative?

Let's Check Your Understanding 6:

The 100th and 200th terms of an arithmetic sequence are 83 and 103, respectively.

a. What is the first term?
b. What is a_{150}?
c. What is a_{345}?

Lesson 7 Solving Problems Involving Arithmetic Means

In an arithmetic sequence, the term(s) between any two terms is (are) called *arithmetic mean***(s)** between two terms.

In the sequence 3, 6, 9, 12, the two arithmetic means between 3 and 12 are 6 and 9.

Examples:

1. Find the arithmetic means between 2 and 8.

Given two arithmetic means there are four terms in all. Assume that $a_1 = 2$ and $a_4 = 8$. Let us have the diagram of the sequence.

2 , __, __, 8
a_1, a_2, a_3, a_4

$$a_n = a_1 + (n - 1)d$$
$$8 = 2 + (4 - 1)d$$
$$8 = 2 + 3d$$
$$8 - 2 = 3d$$
$$6 = 3d$$
$$d = 2$$

Since the first term, a_1, is given and $d = 2$, then it will be easy for us to find the two arithmetic means.

Hence,

$$a_2 = 2 + 2$$
$$a_2 = 4$$

$$a_3 = 4 + 2$$
$$a_3 = 6$$

The numbers 4 and 6 are the two arithmetic means between 2 and 8

2. Find the five arithmetic means between 5 and 47.

Given five arithmetic means there are seven terms in all. Assume that $a_1 = 5$ and $a_7 = 47$. Let us have the diagram of the sequence.

5 , __, __, __, __, __, 47
$a_1, a_2, a_3, a_4, a_5, a_6, a_7$

$$a_n = a_1 + (n - 1)d$$
$$47 = 5 + (7 - 1)d$$
$$47 = 5 + 6d$$
$$47 - 5 = 6d$$

$$42 = 6d$$
$$d = 7$$

Hence, $a_2 = 5 + 7 = \mathbf{12}$
$a_3 = 5 + 14 = \mathbf{19}$
$a_4 = 5 + 21 = \mathbf{26}$
$a_5 = 5 + 28 = \mathbf{33}$
$a_6 = 5 + 35 = \mathbf{40}$

The numbers 12, 19, 26, 33, and 40 are the five arithmetic means between 5 and 47.

3. Insert six arithmetic means between 2 and 16. Also, prove that their sum is 6 times the arithmetic mean between 2 and 16.

2 , __, __, __, __, __, __,16
a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8

Let a_1, a_2, a_3, a_4, a_5, a_6 be the six arithmetic means between 2 and 16. Then, by definition, 2, a_2, ..., a_6, 16 are in arithmetic sequence.

Let d be the common difference. Here, 16 is the 8th term.

$$a_n = a_1 + (n - 1)d$$
$$a_8 = 2 + (8 - 1)d$$
$$16 = 2 + (8 - 1)d$$
$$16 = 2 + 7d$$
$$16 - 2 = 7d$$
$$\frac{14}{7} = d$$
$$d = 2$$

Hence, the six arithmetic means are 4, 6, 8, 10, 12, and 14.

Now the sum of these means: 4 + 6 + 8 + 10+ 12 + 14 = 54.

Find the arithmetic mean between 2 and 16.

Let d be the common difference. Here 16 is the 3rd term.

$$a_n = a_1 + (n - 1)d$$
$$a_3 = 2 + (3 - 1)d$$
$$16 = 2 + (3 - 1)d$$
$$16 = 2 + 2d$$
$$16 - 2 = 2d$$
$$\frac{14}{2} = d$$
$$d = 7$$

Hence, the arithmetic mean between 2 and 16 is 9.

The sum of the 6 arithmetic means between 2 and 16, which is 54 is 6 times its arithmetic mean, 9. That is 54 = 6(9).

Let's Practice for Mastery 7:

Solve what is asked:

1. Insert four arithmetic means between -1 and 14.
2. Insert five arithmetic means between 14 and 86.
3. Insert three arithmetic means between -18 and 4.
4. Insert four arithmetic means between 12 and -3
5. Insert one arithmetic mean between 24 and 68. Such a number is called the *arithmetic mean of the two numbers*.

Let's Check Your Understanding 7:

Solve what is asked:

1. Find the arithmetic mean of 7 and -15.
2. Find the four arithmetic means between 7 and -15.
3. Find the arithmetic mean of $\frac{3}{5}$ and $\frac{5}{3}$.
4. Insert 5 arithmetic means between -2 and 10. Show that their sum is 5 times the arithmetic mean between -2 and 10.
5. Insert 10 arithmetic means between -5 and 17 and prove that their sum is 10 times the arithmetic mean between -5 and 17.

Lesson 8 The Arithmetic Series

Eight basketball teams are participating in the summer sportsfest. If each of the team plays once with each of the others, how many games will be played in all?

Let us simplify the problem by representing the 8 teams by letters A, B, C, …, H. The game played by team A and team B can be represented by a pair of letters, AB. Listing the possible games played,

AB, AC, AD, AE, AF, AG, AH	7 games
BC, BD, BE, BF, BG, BH	6 games
CD, CE, CF, CG, CH	5 games
DE, DF, DG, DH	4 games
EF, EG, EH	3 games
FE, FH	2 games
EH	1 game

The total number of games played by the 7 pairs of teams is 1 + 2 + 3 + 4 + 5 + 6 + 7.

The indicated sum of the terms of this sequence is called a *series*.

The indicated sum played by the 7 pairs of teams is illustrated as:

$S_1 = 1$
$S_2 = 1 + 2$
$S_3 = 1 + 2 + 3$
$S_4 = 1 + 2 + 3 + 4$
$S_5 = 1 + 2 + 3 + 4 + 5$
$S_6 = 1 + 2 + 3 + 4 + 5 + 6$
$S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$

Since, we are interested with the sum of the games played by the 7 pairs of teams then, $S_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$
$S_7 = 28$.

Notice, that the number of games played by the teams form an arithmetic sequence. Thus, the indicated sum of the terms of this arithmetic sequence is called an *arithmetic series* and sum S_n is called the *value* of the series.

In the arithmetic sequence 1, 3, 5, 7, 9, 11, …, the arithmetic series is\

1 + 3 + 5 + 7 + 9 + 11 + 13.

$S_7 = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$

For an arithmetic series in which a_1 is the first term, *d* is the common difference, a_n is the last term, and S_n is the value of the series,

$$S_n = \frac{n(a_1 + a_n)}{2} \text{ and } S_n = \frac{n[2a_1 + (n-1)d]}{2}.$$

Examples:

1. Find the sum of the first 15 odd numbers

   Let $a_1 = 1$, d = 2, and n = 15 in the formula $S_n = \frac{n[2a_1 + (n-1)d]}{2}$

$$S_{15} = \frac{15[2(1) + (15-1)2]}{2}$$
$$= \frac{15[2 + (14)2]}{2}$$
$$= \frac{15[2 + 28]}{2}$$
$$= \frac{15[30]}{2}$$
$$= \frac{450}{2}$$
$$S_{15} = 225$$

2. Find the sum of the first 30 terms of the arithmetic sequence -15, -13, -11,…

Let a_1 = -15, d = 2, and n = 30 in the formula $S_n = \frac{n[2a_1 + (n-1)d]}{2}$

$$S_{30} = \frac{30[2(15) + (30-1)2]}{2}$$
$$= 15[-30 + 2(29)]$$
$$= 15[-30 + 58]$$
$$= 15\,(28)$$
$$S_{30} = 420$$

3. How many numbers between 10 and 200 are exactly divisible by 7? Find their sum.

We know that the first and last number between 10 and 200 which is divisible by 7 is 10 and 196. Hence, a_1 = 10, a_n = 196 and d = 7.

To find n, use $a_n = a_1 + (n – 1)d$

$$196 = 14 + (n - 1)7$$
$$196 – 14 = 7n – 7$$
$$182 = 7n – 7$$
$$189 = 7n$$
$$n = 27$$

There are 27 numbers between 10 and 200 that are divisible by 7.

To find the sum use, $S_n = \frac{n(a_1 + a_n)}{2}$.

$$S_n = \frac{n(a_1 + a_n)}{2}$$
$$S_{27} = \frac{27(14 + 196)}{2}$$
$$= \frac{27(210)}{2}$$
$$= 27(105)$$
$$S_{27} = 2{,}835$$

Let's Practice for Mastery 8:

A. Find the sum of the terms in the arithmetic sequence for the number of terms indicated.

1. 4 + 1 + -2 + -5 + … 40 terms
2. 6 +12 +18 +24 + … 15 terms
3. 10 + 7 + 4 + 1 + … 35 terms
4. 13 + 12 +11 + … 50 terms
5. 2 + 9 +16 + 23+ … 25 terms

B. Find the term asked by using the given values.

1. a_n = 45, a_1 = 27, d = 9, n

2. $a_n = 79$, $a_1 = 7$, $d = 3$, S_n
3. $a_1 = \frac{-3}{4}$, $d = \frac{3}{4}$, $n = 8$, S_8
4. $d = -5$, $a_7 = -11$, $n = 27$, S_{27}
5. $a_{10} = 88$, $a_1 = -8$, S_{10}

Let's Check Your Understanding 8:

Solve the following:

1. Find the sum of the first 150 counting numbers.
2. Find the sum of the first 50 odd natural numbers.
3. Find the sum of the first 42 terms of arithmetic sequence 5, 8, 11, 14, 17, ..., 231
4. How many numbers between 25 and 400 are multiples of 11? Find their sum.
5. Find the sum of all positive integers between 29 and 210 that are divisible by 4.

Lesson 9 Applications in Real Life

A school librarian purchases 10 assorted books during the first month of a contract from a publisher, 15 in the second month, 20 in the third month, 25 in the fourth month, and so on. The librarian wants to know the total number of books the school will have acquired after 30 months. Note that the sequence is 10, 15, 20, 25,…

With $d = 5$, $a_1 = 10$ and $n = 30$, substituting these in the formula $S_n = \frac{n[2a_1 + (n-1)d]}{2}$,

we have

$$S_{30} = \frac{30[2(10) + (30-1)5]}{2}$$
$$= 15\,[20 + (29)5]$$
$$= 15\,[165]$$
$$S_{50} = 2475$$

Thus, after 30 months the school will acquire 2,475 books.

Let's Practice for Mastery 9:

Solve the following problems.

1. If a clock strikes the appropriate number of times on each hour, how many times will it strike in one day? In one week?

2. A group of hikers has a trek of 6 days to reach Mt. Apo. They traveled 15 km on the 1st day, 13 km on the 2nd day, 11 on the 3rd day, and so on. How many kilometers did they travel to reach Mt. Apo?

3. Luis applied for scholarship and was given battery of test. He made a score of 68 on his first test. The passing average score is 75. Would he make it after four test

if he did 6 points better on each succeeding test? What was his score on the fourth test? What was his average score in the battery test?

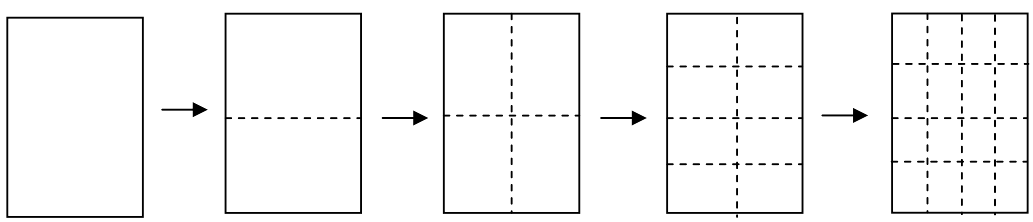
Let's Check Your Understanding 9:

Solve the following problems.

1. Jessie decided to do one more pushup in his exercise each day than he had done the previous day. The first day he did 10 pushups.
   a. How many pushups did Jessie do on the 20^{th} day?
   b. How many pushups did Jessie do altogether in 20 days?
2. Athena vowed to study $\frac{1}{4}$ hr more each day than the previous day. The first day she studied $1\frac{1}{4}$ hr.
   a. How many hours did Athena study on the 10^{th} day?
   b. How many hours did Athena study altogether in 10 days?

Lesson 10 The Geometric Sequence

Making one fold on a sheet of a paper, we can form two rectangles. Now let us fold the paper again, and count the rectangles formed (count only the smallest rectangle as shown below). Continue this process until you can no longer fold the paper.



1 2 4 8 16

The number of rectangles formed produces a geometric sequence,

$$1, 2, 4, 8, 16, \ldots$$

Notice each term after the first may be formed by multiplying the previous term by 2.

A geometric sequence is a set of terms in which each term after the first is obtained by multiplying the preceding term by the same fixed number called the common ratio which is commonly represented by r.

A sequence a_n is called geometric sequence if there is a non-zero number r such that

$$a_n = r \cdot a_1, \quad n > 2.$$

The number r is called the common ratio.

Here are some examples of a geometric sequence.

1. 1, 2, 4, 8,…
2. 9, -27, 81, -343,…
3. .1, .05. 025, .0125
4. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

Each is called a geometric sequence since there is a common multiplier or common ratio, r, between the terms of the sequence after the 1st term. In (1), the common ratio, r, is 2; in (2), r = -3; in (3), r = 0.5: and in (4), r = $\frac{1}{2}$.

The common ratio can be found by dividing any term by its preceding term.

The sequence 2, 6, 18, 54, 162, … is a geometric sequence in which the first term, a_1, is 2 and the common ratio is

$$r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}$$

$$r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$$

Example:

1. Find r and the next three terms of the geometric sequence

$$15, \frac{15}{2}, \frac{15}{4}, \frac{15}{8},\ldots$$

Solution: To find r, choose any two consecutive terms and divide the second by the first.

Choosing the second and third terms of the sequence,

$$r = \frac{a_3}{a_2} = \frac{15}{4} \div \frac{15}{2} = \frac{15}{4} \cdot \frac{2}{15} = \frac{1}{2}$$

To find the next three terms, multiply each successive term by $\frac{1}{2}$

$$a_n = r \cdot a_{n-1}$$

$$a_5 = \frac{1}{2} \cdot a_{5-1} = \frac{1}{2} \cdot a_4 = \frac{1}{2} \cdot \frac{15}{8} = \frac{15}{16}$$

$$a_6 = \frac{1}{2} \cdot a_{6-1} = \frac{1}{2} \cdot a_5 = \frac{1}{2} \cdot \frac{15}{6} = \frac{15}{32}$$

$$a_7 = \frac{1}{2} \cdot a_{7-1} = \frac{1}{2} \cdot a_6 = \frac{1}{2} \cdot \frac{15}{32} = \frac{15}{64}$$

The common ratio is $\frac{1}{2}$ and the next three terms are $\frac{15}{16}, \frac{15}{32}, \frac{15}{64}$.

2. Find the first five terms of a geometric sequence whose first term is 2 and whose common ratio is -3.

Solution: Since $a_1 = 2$ and $r = -3$, then

$$a_1 = 2$$
$$a_2 = r \cdot a_{2-1} = r \cdot a_1 = -3 \cdot 2 = -6$$
$$a_3 = r \cdot a_{3-1} = r \cdot a_2 = -3\,(-6) = 18$$
$$a_4 = r \cdot a_{4-1} = r \cdot a_3 = -3\,(18) = -54$$
$$a_5 = r \cdot a_{5-1} = r \cdot a_4 = -3\,(-54) = 162$$

The first 5 terms of the sequence are 2, -16, 18, -54 and 162.

Let's Practice for Mastery 10.1

A. Tell whether the following sequences is geometric or not. If geometric, find r.

1. 4, 8, 16, 32,…
2. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \ldots$
3. 1, -3, 9, -27, 81,…
4. 1, $\frac{-1}{2}, \frac{-1}{4}, -\frac{-1}{8}, \frac{-1}{16}, \ldots$
5. 5, 15, 45, 135,…

B. Write the first five terms of the geometric sequence where,

1. $a_1 = 2$, $r = 3$
2. $a_1 = 3$, $r = 2$
3. $a_1 = 10$, $r = \frac{1}{2}$
4. $a_1 = 32$, $r = \frac{1}{4}$
5. $a_1 = 3$, $r = -2$

Let's Check Your Understanding 10.1:

A. Tell whether each sequence below is geometric or not. If geometric, find r.

1. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$
2. 1, -3, 7, -11,…
3. $\frac{2}{3}, \frac{2}{15}, \frac{2}{75}, \frac{2}{375}, \ldots$
4. 1, -4, 16, -64,…
5. 10, 20, 30, …

B. Write the first five terms of the geometric sequence where

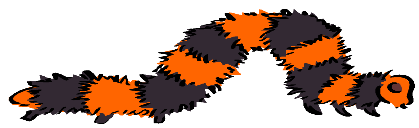
1. $a_1 = 2$, $r = -3$
4. $a_1 = -1$, $r = 0.5$

2. $a_1 = \frac{2}{3}$, $r = -\frac{1}{2}$

5. $a_1 = -2$, $r = -2$

3. $a_1 = 1$, $r = 0.5$

Let's Do It 10.2:



What is the World's Fastest Insect?

Answer the following problems on geometric sequence to find out. Cross out the boxes that contain an answer. The remaining boxes will spell out the name of the world's fastest insect, which can travel at a speed of around 60 kilometers per hour. **Really amazing!** (Source: Math Journal Vol. X, No. 4, 2002 – 2003)

1. What is the common ratio of the geometric sequence 81, 243, 729, . . . ?
2. What is the missing term of the sequence 5, 15, 45, ☐, 405, . . .?
3. What are the next two terms of the sequence 7, 49, 343, . . . ?
4. What is the common ratio of the sequence $\frac{3}{4}, \frac{6}{12}, \frac{12}{36}, \ldots$?
5. Give the next two terms of the sequence $\frac{3}{5}, \frac{6}{25}, \frac{12}{125}, \ldots$.
6. Find the missing term of the sequence $\frac{5}{6}, \frac{10}{18}$, ☐ , $\frac{40}{162}, \ldots$?

B 3	D 9	U $\frac{20}{54}$	R $\frac{3}{2}$	A 2401;16807
T 16	G $\frac{1}{2}$	T $\frac{2}{3}$	R 135	O 27
N 12401; 16807	F $\frac{3}{4}$	E $\frac{24}{625}; \frac{48}{3125}$	L $\frac{20}{56}$	Y 125

Answer: ____ ____ ____ ____ ____ ____ ____ ____ ____

Lesson 11 The nth term of a Geometric Sequence

What if you are asked to find, the 15^{th} term of a geometric sequence? Does it mean that you have to find the 14^{th} term first to get the 15^{th} term? But, since the 14^{th} term is not given, you have to compute the13th term to get the 14^{th} term. In other words, you have to get first all the terms preceding the 15^{th} term.

Is there a shorter way of doing this? Actually, there is! There is a rule or formula for the nth term of any geometric sequence.

> Rule or Formula for the General Term of a Geometric Sequence:
>
> If a_n is a geometric sequence with common ratio, r, then
>
> $$a_n = a_1 \cdot r^{n-1}$$
>
> where n is the number of the term (term number) and a_1 is the first term.

A lot of problems involving geometric sequence is solved using this rule for the general term of a geometric sequence. You will see in the following examples.

Examples:

1. Find the first five terms of a geometric sequence whose first term is 2 and whose common ratio is -3.

   Since $a_1 = 2$ and r = -3, then proceed as follows

   $a_n = a_1 \cdot r^{n-1}$
   $a_2 = a_1 \cdot r^{2-1} = a_1 \cdot r^1 = 2(-3) = -6$
   $a_3 = a_1 \cdot r^{3-1} = a_1 \cdot r^2 = 2(-3)^2 = 2(9) = 18$
   $a_4 = a_1 \cdot r^{4-1} = a_1 \cdot r^3 = 2(-3)^3 = 2(-27) = -54$
   $a_2 = a_1 \cdot r^{5-1} = a_1 \cdot r^4 = 2(-3)^4 = 2(81) = 162$

   The first 5 terms of the sequence are 2, -6, 18, -54, and 162.

The rule for finding the general term of a geometric sequence is a convenient way for you to find the nth term of a geometric sequence. You do not depend on the previous term to get the next term.

2. Find the nth term of the geometric sequence whose first three terms are given below.

   a. $4, \frac{8}{3}, \frac{16}{9}$ b. 5, -10, 20

Solution:

Since the general term of a geometric sequence is $a_n = a_1 \cdot r^{n-1}$, you have to identify the first term and the common ratio.

The first term is:

a. $a_1 = 4$ b. $a_1 = 5$

The common ratio is not given, so you have to find the common ratio by dividing a term by the preceding term. For this case, take a_2 and a_1 so that

a. $r = \frac{a_2}{a_1} = \frac{\frac{8}{3}}{4} = \frac{8}{3} \div 4 = \frac{8}{3} \cdot \frac{1}{4} = \frac{2}{3}$ b. $r = \frac{a_2}{a_1} = \frac{-10}{5} = -2$

Replace a_1 and r into the rule for the general term:

a. $a_n = a_1 \cdot r^{n-1}$

$a_n = 4 \cdot \left(\frac{2}{3}\right)^{n-1}$

b. $a_n = 5 \cdot (-2)^{n-1}$

Notice that it is no longer possible to simplify further

Let's Practice for Mastery 11.1:

A. Write the first five terms of the geometric sequence with the given 1st term and common ratio.

1. $a_1 = 5$ $r = 2$
2. $a_1 = 3$ $r = 4$
3. $a_1 = 3$ $r = -0.5$
4. $a_1 = -3$ $r = -2$
5. $a_1 = .5$ $r = 0.5$

B. Find the nth term of the geometric sequence.

1. 2, 8 , 32,…
2. -4, 12, -36,…
3. 6, 4, $\frac{8}{3}$,…
4. -6, 5, $\frac{-25}{6}$,…
5. 9, -3, 1,…


Let's Check Your Understanding 11.1:

A. Write the first five terms of the geometric sequence with the given first term and common ratio.

1. $a_1 = 3$ $r = 2$
2. $a_1 = 1$ $r = 5$
3. $a_1 = 4$ $r = 0.5$
4. $a_1 = -5$ $r = 2$
5. $a_1 = 0.3$ $r = -0.5$

B. Find the nth term of the geometric sequence.

1. 1, 5, 25,…
2. -3, 6, -12,…
3. $8, 6, \frac{9}{2},\ldots$
4. $-2, \frac{4}{3}, \frac{-8}{9},\ldots$
5. $8, \frac{-4}{3}, \frac{2}{9},\ldots$
6. 0.5, 0.05, 0.005,…

Let's Do It 11.2:

The World's Tallest Skyscraper

The world's tallest skyscraper is 509 meters high. It is found in Taipei and is 101 stories high. It took over the title from the Petronas Twin Towers (452 m) in Kuala Lumpur which was considered the tallest building from 1997 to 2003.

Find the world's tallest skyscraper by answering the following.

Directions: Encircle the letter that corresponds to the correct answer. The letters will spell out the name of the skyscraper.

1. What is the third term of the geometric sequence, $a_n = (-2)^{n-1}$?
   M. 8 T. 4 R. – 8 S. -4

2. What is the second term of the geometric sequence, $a_n = (-1)^{n-1}$?
   A. – 1 H. 0 K. 1 O. 2

3. The fourth term of the geometric sequence, $a_n = 2(3)^{n-1}$ is ____.
   P. 26 S. 36 I. 54 U. 62

4. ______ is the first term of the geometric sequence, $a_n = 3(2)^{n-1}$
   P. 3 S. 6 T. 9 U. 12

5. In $a_n = \frac{1}{4}(4)^{n-1}$, a_4 is equal to _____.

   I. $\frac{1}{4}$ M. 1 N. 4 E. 16

6. In $a_n = \frac{1}{2}(-4)^{n-1}$, 8 is the _____ term.

   K. 1st L. 2nd I. 3rd M. 4th

7. In $a_n = a_1 r^{n-1}$, if $a_1 = 8$ and $r = \frac{2}{3}$, then what is a_6?

   I. $\frac{256}{243}$ B. $\frac{128}{243}$ D. $\frac{64}{243}$ E. $\frac{16}{243}$

8. In $a_n = a_1 r^{n-1}$, find the eighth term of the geometric sequence whose first term is 64 and whose ratio is $-\frac{1}{2}$?

   P. $\frac{1}{8}$ Q. $\frac{1}{4}$ R. $\frac{1}{2}$ O. $-\frac{1}{2}$

9. Find the common ratio of the geometric sequence 81, 54, 36, 24.

   A. $\frac{1}{3}$ I. $\frac{2}{3}$ J. 3 K. 2

Answer: ___ ___ ___ ___ ___ ___   ___ ___ ___

Lesson 12 The General Term of a Geometric Sequence

You are now ready to apply the formula of a geometric sequence. As you go over the examples, the specific skills will be identified.

Examples:

Finding the Specific Term of a Geometric Sequence

1. Find the sixth term of the geometric sequence 3, 6, 12,…

Solution: Find the common ratio.

$$r = \frac{a_2}{a_1} = \frac{6}{3} = 2$$

Substitute in the formula for the nth term of a geometric sequence with n = -6, $a_1 = 3$, r = 2.

$$a_n = a_1 \cdot r^{n-1}$$
$$a_6 = 3 \cdot (2)^{6-1} = 3(2)^5 = 3(32) = 96$$

Notice that the problem simply asks for only the 6th term. This can be solved by listing all the terms of the sequence. Thus, continuing the sequence will give the 6th term.

$$\begin{array}{cccccc} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3, & 6, & 12, & 24, & 48, & 96 \end{array}$$

Thus, the 6th term is 96.

2. Find the 7th term of the geometric sequence whose first term is 4 and whose common ratio is -3.

Solution: Since $a_1 = 4$, $r = -3$ and the 7th term is to be found, use the formula

$$a_n = a_1 \cdot r^{n-1}$$
$$a_7 = a_1 \cdot r^{7-1}$$
$$a_7 = 4(-3)^6$$
$$= 4(729)$$
$$a_7 = 2916$$

Finding a Term, Given Two Other Terms of a Geometric Sequence

3. The third and sixth terms of a geometric sequence are 5 and -4, respectively. Find the eighth term.

Solution: Notice that neither the first term nor the common ratio is given in this problem. So the solution for this one is not the conventional way of solving geometric sequences.

Let $a_3 = 5$ and $a_6 = -40$, $n = 3$ and $n = 6$ in $a_n = a_1 \cdot r^{n-1}$

$a_3 = a_1r^{3-1}$ and $a_6 = a_1r^{6-1}$

$a_3 = a_1r^2$ $\qquad a_6 = a_1r^5$

$5 = a_1r^2$ (1) $\qquad -40 = a_1r^5$ (2)

A system of equations in two variables occurs. Recall that one way of solving a system of linear equations in 2 variables is by substitution.

To do this, solve for one variable, in terms of the other. In this case, solve for a_1 in terms of r, in equation (1).

So that, $5 = ar^2$

$$\frac{5}{r^2} = \frac{a_1 \cdot r^2}{r^2} \qquad \text{Divide both sides by } r^2.$$

$$\frac{5}{r^2} = a_1$$

Then, replace a_1 by $\frac{5}{r^2}$ in equation (2)

$$-40 = a_1 \cdot r^5$$

$$-40 = \left(\frac{5}{r^2}\right) \cdot r^5 \qquad \text{simplifying } \frac{r^5}{r^2} \text{ gives } r^3$$

$$\frac{-40}{5} = \frac{5r^3}{5} \qquad \text{divide both sides by 5}$$

$$-8 = r^3$$

$$r = -2 \qquad \text{since -8 is the third power of -2}$$

Substitute -2 for r in any of the two equations to solve for the other missing variable, a_1. Using equation (1),

$$5 = a_1 r^2 \qquad (1)$$

$$5 = a_1 \cdot (-2)^2$$

$$5 = 4 \cdot a_1$$

$$\frac{5}{4} = \frac{4a_1}{4} \qquad \text{Divide both sides by 4.}$$

$$a_1 = \frac{5}{4}$$

Finally, solve the problem, that is, find the 8th term:

$$a_n = a_1 \cdot r^{n-1}$$

$$a_8 = a_1 \cdot r^{8-1}$$

$$a_8 = \left(\frac{5}{4}\right) \cdot (-2)^7$$

$$a_8 = \left(\frac{5}{4}\right) \cdot (-128) = 5(-32) = -160$$

There is another way of solving the problem above without using systems of equations. It is given below.

Solution 2: Since the geometric sequence gives the 3rd and 6th terms, it can be written as

__, __, 5, __, __, -40, __, __

a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8

Deleting the first two terms, another geometric sequence is found that begins as

5, __, __, -40, __, __

a_1, a_2, a_3, a_4, a_5, a_6

Note that this second geometric sequence has the same common ratio as the original geometric sequence. For this second sequence, $a_1 = 5$ and $a_4 = -40$. Substituting the formula $a_n = a_1 \cdot r^{n-1}$, then

$$a_4 = a_1 \cdot r^{4-1}$$
$$a_4 = a_1 \cdot r^3$$
$$-40 = 5 \cdot (r^3)$$
$$\frac{-40}{5} = \frac{5r^3}{5} \qquad \text{Divide both sides by 5.}$$
$$-8 = r^3$$
$$-2 = r \qquad \text{Take the cube root of each side.}$$

Since what is asked is the 8th term in the original sequence and it has become the 6th term in the second sequence, solve for the 6th term in the second sequence.

$$a_6 = a_1 r^{6-1}$$
$$a_6 = a_1 r^5$$
$$a_6 = 5(-2)^5$$
$$a_6 = 5(-32)$$
$$a_6 = -160$$

Therefore, the 8th term in the original sequence is -160. Notice that you do not have to compute for the first term of the original sequence since what is asked is only to find the 8th term.

Finding the Term Number (Position) of a Term in a Finite Geometric Sequence

4. In the geometric sequence whose first term is -5 and whose common ratio is -2, which term is 10,240?

Solution: Let $a_1 = -5$, $r = -2$ and 10,240 as the nth term in $a_n = a_1 \cdot r^{n-1}$

$$10{,}240 = -5(-2)^{n-1}$$
$$\frac{10{,}240}{-5} = \frac{-5(-2)^{n-1}}{-5}$$
$$-2048 = (-2)^{n-1}$$
$$-2048 = \frac{(-2)^n}{(-2)^1} \qquad \text{Since } a^{n-1} \text{ is the same as } \frac{a^n}{a^1}.$$
$$(-2)\left[-2048 = \frac{(-2)^n}{-2}\right](-2) \qquad \text{Multiply each side by (-2).}$$
$$4096 = (-2)^n$$
$$n = 12 \qquad \text{Since 4096 is the } 12^{th} \text{ root of (-2).}$$

Therefore, 10 240 is the 12th term.

Notice that the terms in a geometric sequence get quite large early in the sequence where the common ratio is greater than 1. The problem above can then be solved by listing down all the terms until the needed term is obtained.

Solution 2: Since the 1st term is -5 and the common ratio is -2, then by listing the terms of the problem above,

-5, 10, -20, 40, -80, 160, -320, 640, -1280, 2560, -5120, 10240
a_1 a_7 a_{12}

It is easy to see that 10,240 is the 12th term.

5. Find the common ratio of a geometric sequence if the first term is $\frac{1}{2}$ and the eighth term is $\frac{2187}{2}$.

Solution:

Let $a_1 = \frac{1}{2}$, and $a_8 = \frac{2187}{2}$ in $a_n = a_1 \cdot r^{n-1}$

$$a_8 = \frac{1}{2} \cdot r^{8-1}$$

$$\frac{2187}{2} = \frac{1}{2} \cdot r^7$$

$$2\left[\frac{2187}{2} = \frac{r^7}{2}\right]2 \qquad \text{Multiply both sides by 2.}$$

$$2187 = r^7$$

$$7 = r \qquad \text{Since } 3^7 = 2187.$$

Therefore, the common ratio is 7.

Let's Practice for Mastery 12:

A. Find the indicated term for each geometric sequence.

1. 2, 10, 50,… a_{10}
2. -1, -3, -9,… a_{15}
3. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$ a_{12}
4. $\frac{2}{3}, \frac{-1}{3}, \frac{1}{6}, \ldots.$ a_{18}
5. $a_1 = 5$, $r = \frac{-1}{5}$ a_{40}

B. Solve as directed.

1. Find the 8th term of the geometric sequence 8, 4, 2, 1,…

2. Find the 6^{th} term of the geometric sequence whose first two terms are 4 and 6.
3. Find the 10^{th} term of the geometric sequence whose 5^{th} term is 48 and 8^{th} term is -384.

Let's Check Your Understanding 12:

A. Find the indicated term for each geometric sequence.

1. 2, 8, 32,…. a_9
2. 4, 3, $\frac{9}{4}$,.... a_8
3. 6, -4, $\frac{8}{3}$,.... a_7
4. -5, 15, -45,…. a_7
5. 1, $\sqrt{2}$, 2,.... a_9

B. Solve as directed.

1. Find the 1^{st} term in the geometric sequence where the 4^{th} term is 4 and the 7^{th} term is 32.

2. In the geometric sequence 4, 64, 1 024,…, which term is 262 144?

Lesson 13 Geometric Means

When the first and the last terms of a geometric sequence are given, the terms between them are called the geometric means. For example, the geometric means of the geometric sequence 2, 6, 18, 54, 162 are *6, 18* and *54*.

To solve for the geometric means of a given geometric sequence, the formula for the nth term of a geometric sequence is also used.

Example:

1. Insert 3 geometric means between 4 and 324.

Solution: Listing down the geometric sequence will show that there are five terms, which means that n = 5. So that $a_5 = 324$ and $a_1 = 4$.

4, __, __, __, 324

Substituting in the formula:

$$a_n = a_1 \cdot r^{n-1}$$

$$a_5 = 4 \cdot r^{5-1}$$

$$324 = 4 \cdot r^4$$

$$\frac{324}{4} = \frac{4 \cdot r^4}{4} \qquad \text{divide both sides by 4}$$

$$81 = r^4$$

$$\pm 3 = r \qquad \text{since 81 is the fourth power of } \pm 3$$

Note that the common ratio, r, takes two values, +3 and -3. So that there are two sets of geometric means that can answer the question. To get the desired geometric means simply multiply the preceding terms by the common ratio.

For r = 3, 4, 12, 36, 108, 324

For r = -3, 4, -12, 36, -108, 324

Therefore, the geometric means are ±12, 36, and ±108.

2. Insert four geometric means between 3 and 96.

Solution: Listing down the terms of the sequence gives

3, __, __, __, __, 96

Let n = 6, a_6 = 96 and a_1 = 3 in $a_n = a_1 \cdot r^{n-1}$

$$a_6 = a_1 \cdot r^{6-1}$$

$$96 = 3 \cdot r^5$$

$$\frac{96}{3} = \frac{3 \cdot r^5}{3} \qquad \text{divide both sides by 3}$$

$$32 = r^5$$

$$2 = r \qquad \text{since 32 is the fifth power of 2}$$

Since r is 2 then the answer is shown below.

3, 6, 12, 24, 48, 96

The geometric means are 6, 12, 24 and 48.

3. Find the geometric mean between 12 and 192.

Solution: Here only a single term is asked. So that

12, __, 192

Using the formula for the nth term of a geometric sequence,

$$a_n = a_1 \cdot r^{n-1}$$

$$192 = 12 \cdot r^{3-1}$$

$$192 = 12 \cdot r^2$$

$$\frac{192}{12} = \frac{12 \cdot r^2}{12}$$ divide both sides by 12

$$16 = r^2$$

$$\pm 4 = r$$ 16 is the second power of ±4

Multiplying the first term, 12, by the common ratio, ±4, the computed geometric mean is either 48 or -48.

> Geometric Mean between Two Numbers
>
> If b, c and d form a geometric sequence, then c is the geometric mean between b and d. So that,
>
> $$\frac{c}{b} = \frac{d}{c} \qquad c^2 = bd \qquad c = \pm\sqrt{b \cdot d}$$
>
> Therefore, the geometric mean between two terms / numbers is the square root of the product of the two terms/ numbers.

4. Find the geometric mean between the two numbers.

   a. 8 and 72   b. -7 and -112

Solution: Substituting in the formula for the geometric mean between two numbers

$$c = \pm\sqrt{b \cdot d}$$

a. $c = \pm\sqrt{8 \cdot 72} = \pm\sqrt{576} = \pm 24$

b. $c = \pm\sqrt{(-7)(-112)} = \pm\sqrt{784} = \pm 28$

Let's Practice for Mastery 13:

A. Find the geometric mean between the given two numbers.

1. 4 and 20
2. 8 and 128
3. 2 and $\frac{3}{4}$
4. 6 and 216
5. 4 and 144

B. Do as directed.

1. Insert two geometric means between 15 and $\frac{15}{8}$.

2. Insert four geometric means between 4 and -972.
3. Insert four geometric means between $\frac{25}{4}$ and $\frac{8}{125}$.

Let's Check Your Understanding 13:

A. Find the geometric mean between the given two numbers.

1. $\frac{3}{2}$ and $\frac{25}{25}$
2. -2 and $\frac{-2}{3}$ -
3. $\sqrt{2}$ and $3\sqrt{2}$
4. 30 and 240
5. $2\frac{3}{5}$ and $\frac{20}{7}$

B. Do as directed.

1. Find the geometric mean between 5 and 500.
2. Insert three geometric means between $\frac{2}{3}$ and $\frac{27}{8}$.

Lesson 14 Geometric Series

A frog leaps $\frac{2}{3}$ of the previous jump.

If the frog's first leap is 27cm, find the distance the frog has covered after 5 leaps.

To find the answer to this one, you have to go over with this lesson.

For this lesson, only finite geometric series will be discussed. Infinite geometric series are discussed in the next lesson.

The indicated sum of the terms of a geometric sequence is called a "geometric series," it is denoted by S_n. In symbols,

$$S_n = a_1 + a_1r^1 + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-2} + a_1r^{n-1}$$

The sum of n terms of a geometric sequence is given by:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

where, a_1 = first term of a geometric sequence and r = common ratio, $r \neq 1$

It is good to note that r should not be equal to 1 since if it is, the denominator will be zero and will not make any sense.

But what if r = 1, does it mean that a sum does not exist? Of course, the sum exists. Find out using the concepts you will learn in this lesson.

Examples:

1. Find the sum of the first six terms of the geometric sequence 3, 6, 12, 24,…

Solution: The common ratio is $\frac{6}{3} = 2$. The sum of 6 terms is given by:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_6 = \frac{3(1-2^6)}{1-2}$$

$$= \frac{3(1-64)}{-1}$$

$$= \frac{3(-63)}{-1}$$

$$S_6 = 189$$

2. Find the sum of 10 terms of the sequence: 1, $\frac{1}{2}$, $\frac{1}{4}$,…

Solution: In this case: $a_1 = 1$, $r = \frac{1}{2}$, and $n = 10$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{1\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}}$$

$$S_{10} = \frac{\left[1-\left(\frac{1}{1024}\right)\right]}{\frac{1}{2}}$$

$$S_{10} = \left[\frac{1024}{1024} - \frac{1}{1024}\right] \div \frac{1}{2}$$

$$S_{10} = \frac{1023}{1024} \cdot \frac{2}{1}$$

$$S_{10} = 1.998$$

Using the formula for a geometric series may seem to be tedious but with practice and a little patience, it will turn out to be very easy.

3. Find the sum of the indicated number of terms in the given geometric sequence.

a. $a_1 = 3, r = -1, n = 9$
b. $a_1 = 8, r = -1, n = 51$
c. $a_1 = 3, r = -1, n = 12$
d. $a_1 = 8, r = -1, n = 30$

Solution:

a. $S_n = \frac{a_1(1-r^n)}{1-r} = S_9 = \frac{3[1-(-1)^9]}{1+1} = \frac{3[1-(-1)]}{2} = \frac{3[2]}{2} = 3$

b. $S_n = \frac{a_1(1-r^n)}{1-r} = S_{51} = \frac{8[1-(-1)^{51}]}{1+1} = \frac{8[1-(-1)]}{2} = \frac{8[2]}{2} = 8$

c. $S_n = \frac{a_1(1-r^n)}{1-r} = S_{12} = \frac{3[1-(-1)^{12}]}{1+1} = \frac{3[1-1]}{2} = \frac{3[0]}{2} = 0$

d. $S_n = \frac{a_1(1-r^n)}{1-r} = S_{30} = \frac{8[1-(-1)^{30}]}{1+1} = \frac{3[1-1]}{2} = \frac{8[0]}{2} = 0$

From Example 3, one can generalize that if r = -1, then

$S_n = a_1$ when n is odd or
$S_n = 0$ when n is even.

4. Find the sum of the geometric series: $3 + 12 + 48 + \ldots$ up to 5 terms

Solution: Using the formula for the sum of a geometric series with $a_1 = 3$ and $r = 4$.

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{3[1-(4)^5]}{1-4} = \frac{3[1-1024]}{-3} = \frac{3(-1023)}{-3} = 1023$$

The sum is 1023.

Let's Practice for Mastery 14:

A. Do as directed.

1. Find the sum of the first 8 terms of the geometric sequence: 2, 4, 8, 16,…
2. What is the sum of the first 3 terms of the geometric sequence.
$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots ?$$
3. What is the sum of the first 8 terms of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots ?$

4. Find the sum of the first 4 terms of the geometric series:

$$1 + 10 + 100 + 1000 + \ldots$$

5. Find the sum of the first 7 terms of the geometric sequence: 3, 9, 27, 81,…

B. Fill in the table with the values that will make each a geometric sequence.

No.	a_1	r	n	a_n	S_n
1.	3	-4	8		
2.	2	$\frac{1}{3}$	7		
3.	32		8	$\frac{1}{4}$	
4.		-3	7	-2 916	
5.	$\frac{1}{2}$	$\frac{3}{2}$		$\frac{243}{64}$	

Lesson 15 Sum of an Infinite Geometric Sequence

Sum of an infinite geometric sequence! Is there such a thing? Well, there is! You are actually going to learn it in this lesson.

As an introduction, let us start from what we know. Earlier, we learned that the formula for the sum of a finite geometric series is

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

For example, in the sequence is 6, 12, 24, …

Since r = 2, as n increases, the value of r^n also increases and so does the sum, S_n. Each new term adds a larger and larger amount to the sum and so there is no limit to the value of S_n and S_α does not exist. A similar situation occurs if r = 1, so that generally the sum to infinity of a geometric sequence is

$$S_\alpha = \frac{a_1}{1-r}$$

> The sum of the terms of an infinite geometric sequence with first term a and common ratio r, where $|r| < 1$, is
>
> (1) $S_\alpha = \frac{a_1}{1-r}$ or (2) $S_\alpha = a_1\left(\frac{1}{1-r}\right)$

Examples:

1. Find the sum to infinity of the geometric sequence with $a_1 = 5$ and $r = -\frac{1}{3}$.

Solution: Substituting the given values to the formula above, the sum is

$$S_\alpha = \frac{a_1}{1-r} = \frac{5}{1-(-\frac{1}{3})}$$

$$= \frac{5}{1+\frac{1}{3}} = \frac{5}{\frac{4}{3}}$$

$$= 5 \div \frac{4}{3} = 5 \cdot \frac{3}{4}$$

$$S_n = \frac{15}{4}$$

2. Find the sum to infinity of the geometric sequence 20, 5, $\frac{5}{4}, \frac{5}{16}, ...$

Solution: The common ratio is $\frac{1}{4}$ and $a_1 = 20$. Substituting in the formula

$$S_\alpha = \frac{a_1}{1-r}$$

$$S_\alpha = \frac{20}{1-\frac{1}{4}} = \frac{20}{\frac{3}{4}}$$

$$= 20 \div \frac{3}{4} = 20 \cdot \frac{4}{3}$$

$$= \frac{80}{3}$$

For the next example, the second formula will be used.

3. Find the sum to infinity of the geometric sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$

Solution: Substituting in the second formula, where $a_1 = \frac{1}{3}$ and $r = \frac{1}{3}$,

then $$S_\alpha = a_1\left(\frac{1}{1-r}\right)$$

$$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1}{\frac{2}{3}}\right) = \frac{1}{3}\left(1 \div \frac{2}{3}\right)$$

$$= \frac{1}{3}\left(1 \cdot \frac{3}{2}\right) = \frac{1}{2}$$

Now, look at the solution using the first form of the formula:

$$S_{\alpha} = \left(\frac{a_1}{1-r}\right)$$

$$= \left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

A thorough knowledge of all skills related to fractions helps in the understanding of how the solution is done.

Let's Practice for Mastery 15

Find the sum to infinity of the geometric sequences given below.

1.	12, 4, $\frac{4}{3}$,…	6.	a_1 = 1 000, r = 0.01
2.	900, 9, 0.09,…	7.	36, 24, 16,…
3.	18, 6, 2,…	8.	a_1 = 81, r = 0.1
4.	a_1 = 32, r = $-\frac{1}{2}$	9.	a_1 = 10, r = $\frac{1}{5}$
5.	16, 4, 1,…	10.	$a_1 = \frac{9}{8}$, $r = \frac{-2}{3}$

Lesson 16 Applications of Geometric Sequences and Series

A lot of problems can be solved by using the formulas for the general term of a geometric sequence and geometric series, finite or infinite. Of these applications, that of the infinite geometric series is most interesting as seen in the following examples.

Examples:

A. Changing Repeating Decimals to Fractions:

1. Show that the repeating, non- terminating decimal 0.2727… is equal to $\frac{3}{11}$.

Solution: The decimal can be written as 0.27 + 0.0027 + 0.000027 + …

Writing the decimal as a fraction gives $\frac{27}{100} + \frac{27}{10000} + \frac{27}{1000000} + ...$

The series of numbers really is an infinite geometric series, since there is a common ratio, r = $\frac{1}{100}$, with $a_1 = \frac{27}{100}$. So solving for the sum, gives

$$S_\alpha = \frac{a_1}{1-r} = \frac{\frac{27}{100}}{1-\frac{1}{100}} = \frac{\frac{27}{100}}{\frac{100}{100}-\frac{1}{100}} = \frac{\frac{27}{100}}{\frac{99}{100}}$$

$$= \frac{27}{100} \div \frac{99}{100} = \frac{27}{100} \cdot \frac{100}{99} = \frac{27}{99} = \frac{3}{11}$$

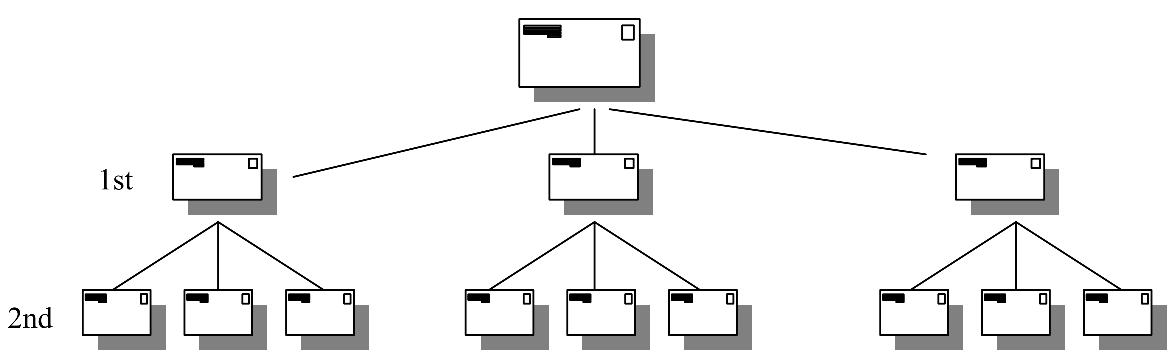
Hence, $0.2727\ldots = \frac{3}{11}$

B. Chain Letter Problem

2. Linda wrote a letter and sends it to three friends. Each of the three friends writes the same letter and sends it to 3 other friends and the sequence is repeated. Assuming that no one breaks the chain, how many letters will have been sent from the first through the sixth mailings?

Solution:

The diagram will help in understanding the problem.



On the first mail, 3 letters are sent, on the second mailing there are 3(3) = 9 letters sent, on the third mailing there are 9(3) = 27 letters sent, and so on. Observe that the sequence formed is 3, 9, 27,…

The problem asked for the total number of letters mailed. So the formula for the sum of n terms of a geometric sequence is used.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_6 = \frac{3(1-3^6)}{1-3}$$

$$= \frac{3(1-729)}{-2}$$

$$= \frac{3(-728)}{-2}$$

$$= \frac{-2184}{-2}$$

$$S_6 = 1092$$

There are 1092 letters mailed in all.

C. Growth of Bacteria:

3. A certain culture of bacteria initially contains 1 000 bacteria and doubles every hour. How many bacteria are in the culture at the end of 10 hours?

Solution: Since the number of bacteria doubles every hour and there are initially 1000, then at the end of the first hour there will be 2000. At the end of the second hour, there will be 4 000 and so on. A table of values is shown below.

t hours	1	2	3	4	5
no. of bacteria	2 000	4 000	8 000	16 000	32 000

The second row of the table shows a geometric sequence where a_1= 2000 and r = 2. Using the formula for the nth term of a geometric sequence, then,

$$a_n = a_1 \cdot r^{n-1}$$
$$= 2000(2)^{10-1}$$
$$= 2000(2)^9$$
$$= 2000(512)$$
$$a_n = 1\ 024\ 000$$

There are 1 024 000 bacteria at the end of 10 hours.

Notice that we did not start the sequence with 1000 since it is the initial number of bacteria in the culture at t = 0. The doubling starts at the end of the first hour.

Let's Practice for Mastery 16:

A. Write each of the following repeating decimals as an equivalent fraction:

1. 0.555…
2. 0.06262…

B. Solve the following.

1. On the first swing, the length of the arc through which a pendulum swings is 20dm. The length of each successive swing is $\frac{4}{5}$ of the preceding swing. What is the total distance traveled by the pendulum has traveled during the four swings?

2. What distance will a golf ball travel if it is dropped from a height of 72 dm, and if, after each fall, it rebounds $\frac{2}{3}$ of the distance it fell?

3. A culture of bacteria doubles every 3 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

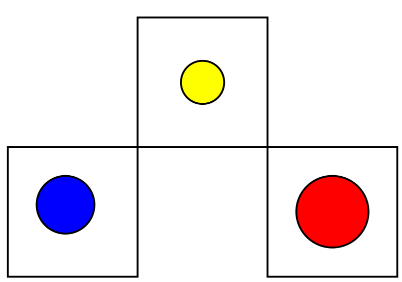
4. A particular substance decays in such a way that it loses half its weight each day. If initially there are 256 grams of the substance, how much is left after 10 days?

C. The following is the Tower of Hanoi Puzzle. Read it and try to do what you are asked. Then answer the questions that follow.

The Tower of Hanoi is a puzzle that has the following form: Three pegs are placed in a board. A number of disk graded in size are staked in one of the pegs with the largest disk at the bottom and the succeeding smaller disk placed on top. The disks are moved according to the following rules:

a. Only one disk at a time may be moved.
b. A larger disk cannot be placed over a smaller disk.

The object of the puzzle is to transfer all the disks from one peg to one of the other two pegs. If initially there is only one disk, then there will be only one move. With three disks, then only one move would be required.



You can try this puzzle using 3 coins of different sizes.

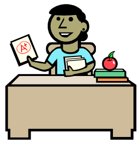
The chart below shows the minimum number of moves required for an initial number of disks. The difference between the numbers of moves for each succeeding disk is also given.

No. of disks	1	2	3	4	5	6	7	8
No. of moves	1	3	7	15	31	63	x	y

2 4 8 16 p q r

Questions:

1. What kind of sequence is the last list of numbers in the chart?
2. Find the values of p, q and r.
3. Then find x and y.
4. What is the general term for the sequence of numbers in the second row?



LET'S SUMMARIZE

A ***sequence*** is a set of numbers written in a specific order:

$$a_1, a_2, a_3, a_4, a_5, a_6, \ldots\ldots\ldots, a_n.$$

where, a_1 is called the 1st term, a_2 is the 2nd term, and in general, a_n is the nth term.

Arithmetic sequence is a sequence where each succeeding term is obtained by adding a fixed number. Common difference *d* is the fixed number between any two succeeding terms.

The terms of an arithmetic sequence are defined by using the formula $a_n = a_1 + (n - 1)d$.

Arithmetic series is an indicated sum of the first n terms of an arithmetic sequence. The sum of n terms is denoted by S_n.

For an arithmetic series in which a_1 is the first term, *d* is the common difference, a_n is the last term, and S_n is the sum of the series,

$$S_n = \frac{n(a_1 + a_n)}{2} \text{ and } S_n = \frac{n[2a_1 + (n-1)d]}{2}.$$

A sequence a_n is called geometric sequence if there is a non-zero number r such that $a_n = r \cdot a_{n-1}$, $n > 2$, such that the number r is called the common ratio.

If a_n is a geometric sequence with common ratio, r, then $a_n = a_1 \cdot r^{n-1}$,where n is the number of the term (term number) and a_1 is the 1st term.

If b, c and d form a geometric sequence then c is the geometric mean between b and d. Thus,

$$\frac{c}{b} = \frac{d}{c} \qquad c^2 = bd \qquad c = \pm\sqrt{b \cdot d}$$

The indicated sum of a geometric sequence is called a geometric series.

The sum of n terms of a geometric sequence or the sum of a geometric series is given by the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where a_1= the first term, n = the number of terms and r = the common ratio.

The sum of an infinite geometric sequence or of an infinite geometric series is given by the formula

$$S_\alpha = \frac{a_1}{1 - r} = a_1\left(\frac{1}{1 - r}\right)$$

where a_1 = the first term and r = the common ratio such that $|r| < 1$.

Unit Test

Answer the following:

1. Write the first five terms of the sequence $a_n = 5n - 2$.
2. Find a_8 in $a_n = -7 + 3$.
3. What is the general term for 0, -4, -8, -12?
4. For the sequence denoted by $a_n = \frac{2n}{5+n}$.
5. Find the first three terms of the sequence $a_n = 3 + 3^n$.
6. Give the arithmetic sequence whose 7th term is 23 and whose 12th term is 38.
7. Find the three arithmetic means between 9 and 33.
8. Find the 25th term of the arithmetic sequence 2, 5, 8, 11, ….
9. Find the arithmetic mean of $\frac{2}{3}$ and $\frac{3}{2}$.
10. Find the sum of the first 30 terms in the arithmetic sequence 0, 1, 2, 3, …
11. What is the sum of the numbers from 1 to 100?
12. How many numbers between 200 and 400 are divisible by 15?
13. Which of the following is a geometric sequence?
14. Find the next four terms in 10, 2, $\frac{2}{5}$, $\frac{2}{25}$, …
15. Find the common ratio of the geometric sequence 3, 6, 12, 24, …
16. What is the general term of the sequence in number 15?
17. Find the 8th term of the sequence 2, 6, 18, …
18. Find the next six terms of the geometric sequence whose $a_1 = 25$ and $r = -\frac{1}{5}$.
19. Insert two geometric means between 28 and 224.
20. Find the sum of the first 12 terms of the geometric sequence 3, 6, 12. …
21. Find an equivalent fraction of the repeating decimal 0.3838, …
22. Rosel started a chain of inspirational text messages to two of her friends. Each of the two friends sends the message to two other friends and the sequence continues. About how many text messages would have been sent after the sixth sending?
23. A particular substance decays in such a way that it loses half of its weight each day. If initially there are 256 grams of the substance, how much is left after 10 days?

24. Marc planned to on a holiday in December at one of the beach resort in Boracay. He started to save Php100 during the month of March and each month thereafter, doubles the amount he saved the month before. How much would have been saved by December?

25. Edna needs Php8000.00 to buy a bicycle. She has already saved Php250. If she saves Php100.00 a week from her job, in how many weeks must she work to have enough money to buy the bicycle?

ANSWER KEY

Lesson 1:

Let's Practice for Mastery 1

1. F
2. F
3. I
4. F
5. F

Let's Do It

Because he can hold up traffic

Let's Check Your Understanding 1

A.

1. F
2. I
3. I
4. F
5. F

B.

1. 162
2. 2187
3. -32
4. -5
5. $\frac{1}{128}$

Lesson 2:

Let's Practice for Mastery 2

A.

1. 2, 3, 4, 5
2. 2, 5, 8, 11
3. 2, 4, 8, 16
4. 2, 5, 10, 17
5. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

B.

1. 40
2. -14
3. 110
4. 225
5. $\frac{1}{256}$

Let's Check Your Understanding 2

A.

1. 0, 1, 2, 3
2. 0, 3, 8, 15
3. 3, 9, 27, 81
4. -1, -3, -5, -7
5. $0, \frac{7}{2}, \frac{26}{3}, \frac{63}{4}$

B.

1. 15
2. 11
3. $\frac{12}{13}$
4. $\frac{32}{243}$
5. 380

Lesson 3:

Let's Practice for Mastery 3

A.
1. $a_n = 3n$
2. $a_n = 3n^2$
3. $a_n = (-2)^n$
4. $a_n = 5n-1$
5. $a_n = \frac{1}{2^{n+1}}$
6. $a_n = \frac{n}{(n+1)^2}$

B.
1. 14
2. 1000
3. 45
4. 49
5. 67

Let's Check Your Understanding 3

A.
1. $a_n = 3n + 4$
2. $a_n = 4n$
3. $a_n = n^2$
4. $a_n = 2n^3$
5. $a_n = \frac{1}{3^n}$
6. $a_n = \frac{n}{3^n + 1}$

B.
1. 8
2. 28
3. 128
4. -7
5. -300

Lesson 4:

Let's Practice for Mastery 4.1
1. Arithmetic; d = 3; 14, 17, 20
2. No
3. Arithmetic; d = -4; -22, -26, -30
4. Arithmetic; d = 2; 48, 50, 52
5. Arithmetic; d = 0.6; 3, 3.6, 4.2
6. Arithmetic; d = 4; 17, 21, 25
7. No
8. No
9. Arithmetic; d = -3; 86, 83, 80
10. Arithmetic; d = $\frac{1}{3}$; $\frac{7}{3}$, $\frac{8}{3}$, 3

Let's Check Your Understanding 5
1. d = 9; 37, 46, 55
2. d = 1.5; 11.5, 13, 14.5
3. d = $\sqrt{3}$ $5\sqrt{3}$, $6\sqrt{3}$, $7\sqrt{3}$
4. d = 2; 9, 11, 13
5. d = -4; 31, 27, 23
6. d = 9; 61, 70, 79

Let's Practice for Mastery 4.2
1. $a_{15} = 30$
2. $a_{25} = 85$
3. $a_{18} = -88$

4. $a_{12} = \frac{5}{2}$
5. $a_{12} = 33$

Let's Check Your Understanding 4.2

1. $a_{23} = 102$
2. $a_{17} = -21$
3. $a_{14} = 14\sqrt{3}$
4. $a_{10} = 106$
5. $a_{22} = -116$

Lesson 5:

Let's Practice for Mastery 5

1. 305.9 dm
2. Plan B; Plan A = P104,000; Plan B = P402,000
3. 1.5 points

Let's Check Your Understanding 5

a. $a_{10} = 232$
b. $a_{20} = 252$
c. ticket no. 94
d. 725 customers

Lesson 6:

Let's Practice for Mastery 6

a. $d = \frac{3}{2}$
b. $a_1 = -\frac{5}{2}$
c. $a_{234} = \frac{694}{2}$
d. 2 terms

Let's Check Your Understanding 6

c. $d = 63$
d. $a_{50} = \frac{9217}{99}$
c. $a_{345} = \frac{13117}{99}$

Lesson 7:

Let's Practice for Mastery 7

1. 2, 5, 8, 11
2. 26, 38, 50, 62, 74
3. $-\frac{25}{2}, -7, \frac{-3}{2}$
4. 9, 6, 3, 0
5. 46

Let's Check Your Understanding 7

1. -4
2. $-\frac{13}{5}, -\frac{9}{2}, -\frac{31}{5}, -\frac{53}{5}$
3. $-\frac{17}{5}$
4. The arithmetic means between -2 and 10 are 0, 2, 4, 6, 8 and the sum is 20. The arithmetic mean between -2 and 10 is 4. 5(4) = 20
5. The 10 arithmetic means between -5 and 17 are -3, -1, 1, 3, 5, 7, 9, 11, 13, 15 and the sum is 60. The arithmetic mean between -5 and 17 is 6. 10(6) = 60

Lesson 8:

Let's Practice for Mastery 8

A.

1. S_{40} = -2,180
2. S_{15} = 720
3. S_{35} = -1,435
4. S_{50} = -575
5. S_{25} = -2,150

1. n = 3
2. S_n = 1075
3. S_8 = 15
4. S_{27} = -1,242
5. S_{10} = 400

Let's Check Your Understanding 8

1. S_{150} = 11,325
2. S_{50} = 2,500
3. S_{42} = 4,956
4. n = 34; S_{34} = 7293
5. n = 45; S_{45} = 5400

Lesson 9:

Let's Practice for Mastery 9

1. 156 strikes in a day; 1092 strikes in a week
2. 60 km
3. yes; a_4 = 86; average = 77

Let's Check Your Understanding 9

1. a. d = 29
   b. s_{20} = 195
2. a. a_{10} = 3.5 hrs
   b. $s_{10} = 18\frac{3}{4}$ hrs

Lesson 10:

Let's Practice for Mastery 10.1

A.

1. Geometric; r = 2
2. No

B.

1. 2, 6, 18, 54, 162
2. 3, 6, 12, 24, 48

3. Geometric; r = -3
4. No
5. Geometric; r = 3

3. 10, 5, $\frac{5}{2}$, $\frac{5}{4}$, $\frac{5}{8}$
4. 32, 8, 2, $\frac{1}{2}$,$\frac{1}{8}$
5. 3, -6 12, -24, 48

Let's Check Your Understanding 10.1

A.

1. No
2. No
3. Geometric; r = $\frac{1}{5}$
4. Geometric; r = -4
5. No

B.

1. 2, -6, 18, -54, 162
2. $\frac{2}{3},-\frac{2}{6},\frac{2}{12},-\frac{2}{24},\frac{2}{48}$
3. 1, 1.5, 2, 2.5, 3
4. -1, -0.5, 0, 0.5, 1
5. -2, 4, -8, 16, -32

Let's Do It 10.2

1. r = 3
2. 135
3. 2401; 16,807
4. r = $\frac{2}{3}$
5. $\frac{24}{625}$, $\frac{48}{3125}$
6. $\frac{24}{54}$

Answer: <u>D</u> <u>R</u> <u>A</u> <u>G</u> <u>O</u> <u>N</u> <u>F</u> <u>L</u> <u>Y</u>

Lesson 11:

Let's Practice for Mastery 11.1

A.

1. 1, 5, 10, 20, 40, 80
2. 3, 12, 48, 192, 768
3. 3, -1.5, .75, -.375, .1875
4. -3, 6, -12, 24, -48
5. 0.5, .25, .125, .0625, .03125

B.

1. $a_n = (2)^{2n-1}$
2. $a_n = -4(-3)^{n-1}$
3. $a_n = 6(\frac{2}{3})^{n-1}$
4. $a_n = -6(\frac{-5}{6})^{n-1}$
5. $a_n = (-\frac{1}{3})^{n-3}$

Let's Check Your Understanding 11.1

A.

1. 3, 6, 12, 24, 48
2. 1, 5, 25, 125, 625

1. $a_n = (5)^{n-1}$
2. $a_n = -3(2)^{n-1}$

3. 4, 2, 1, 0.5, 0.25

4. -5, -10, -20, -40, -80

5. 0.3, -.15, .75, -.375, -.1875

3. $a_n = 8(\frac{3}{4})^{n-1}$

4. $a_n = -2(\frac{-2}{3})^{n-1}$

5. $a_n = 8(-\frac{1}{6})^{n-1}$

6. $5(0.1)^n$

Let's Practice for Mastery 11.2

1. P 4
2. E -1
3. T 54
4. R 3
5. O 16
6. N 3^{rd}
7. A $\frac{256}{243}$
8. S $-\frac{1}{2}$

Answer: TAIPEI 101

Lesson 12:

Let's Practice for Mastery 12

A.

1. $2(5)^9$
2. $-1(3)^{14}$
3. $\frac{1}{2}(\frac{1}{3})^{14}$
4. $-\frac{1}{3(2)^{16}}$
5. $-\frac{1}{5^{38}}$

B.

1. $a_8 = \frac{1}{16}$
2. $a_6 = \frac{243}{8}$
3. $a_{10} = -1536$

Let's Check Your Understanding 12

A.

1. 131072
2. $\frac{2187}{4096}$

3. $\frac{128}{243}$
4. -3645
5. 16

B.

1. $\frac{1}{2}$
2. 5^{th} term

Lesson 13:

Let's Practice for Mastery 13

A.

1. $\pm 4\sqrt{5}$
2. ± 32
3. $\pm \frac{\sqrt{6}}{2}$
4. ± 36
5. ± 24

B.

1. $\frac{15}{2}, \frac{15}{4}$
2. -12, 36, -108, 324
3. $\frac{5}{2}, 1, \frac{2}{5}, \frac{4}{25}$

Let's Check Your Understanding 13

A.

1. $\pm \frac{\sqrt{6}}{2}$
2. $\pm \frac{2\sqrt{3}}{3}$
3. $\pm \sqrt{6}$
4. $\pm 60\sqrt{2}$
5. $\pm \frac{2\sqrt{27}}{7}$

B.

1. 50
2. $1, \frac{3}{2}, \frac{9}{4}$

Lesson 14:

Let's Practice for Mastery 14

A.

1. 510
2. $1\frac{3}{4}$
3. $\frac{255}{128}$
4. 1111
5. 3279

B.

1. $a_n = -\frac{49}{52}$; $s_n = -39321$
2. $a_n = \frac{2}{729}$; $s_n = \frac{2186}{729}$
3. $r = \frac{1}{2}$; $s_n = \frac{255}{4}$
4. $a_1 = 4$; $s_n = -2188$
5. $n = 6$; $s_n = \frac{665}{64}$

Let's Practice for Mastery 15

1. 18
2. $\frac{10000}{11}$
3. 27
4. $\frac{64}{3}$
5. 3
6. $\frac{100,000}{99}$
7. 108
8. 90
9. $\frac{25}{2}$
10. $\frac{27}{88}$

Lesson 16:

Let's Practice for Mastery 16

A.

1. $\frac{5}{9}$
2. $\frac{31}{495}$

B.

1. 100 dm
2. 360 dm
3. 128,000 bacteria
4. $\frac{1023g}{w}$

C.

1. Geometric
2. 32, 64, 128
3. 127, 255
4. $a_n = (2)^n - 1$

Unit Test:

1. 3, 8, 13, 18, 23
2. $a_8 = -53$
3. $a_n = 4(1 - n)$
4. $\frac{2}{8}, \frac{4}{7}, \frac{6}{8}, \frac{4}{9}, \frac{5}{10}$
5. 5, 9, 13, 17
6. 5, 8, 11, 14, 17
7. 15, 21, 27
8. 74
9. $\frac{13}{12}$
10. 435
11. 5050
12. 13
13. b, e
14. $\frac{2}{125}, \frac{2}{625}, \frac{2}{3125}, \frac{2}{15625}$
15. 2
16. $a_n = 3(2)^{n-1}$
17. 4, 374
18. 25, -5, 1, $-\frac{1}{5}, \frac{1}{125}, -\frac{1}{125}$
19. 56 and 112
20. 12285
21. $\frac{38}{99}$
22. 126
23. $\frac{1}{2}$
24. Php51,200.00
25. 27 weeks